

THE FROBENIUS PROPERAD IS KOSZUL

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ABSTRACT. We show Koszulness of the prop governing involutive Lie bialgebras and also of the props governing non-unital and unital-counital Frobenius algebras, solving a long-standing problem. This gives us minimal models for their deformation complexes, and for deformation complexes of their algebras which are discussed in detail.

Using an operad of graph complexes we prove, with the help of an earlier result of one of the authors [W3], that there is a highly non-trivial action of the Grothendieck-Teichmüller group GRT_1 on (completed versions of) the minimal models of the properads governing Lie bialgebras and involutive Lie bialgebras by automorphisms. As a corollary one obtains a large class of universal deformations of any (involutive) Lie bialgebra and any Frobenius algebra, parameterized by elements of the Grothendieck-Teichmüller Lie algebra.

We also prove that, for any given homotopy involutive Lie bialgebra structure in a vector space, there is an associated homotopy Batalin-Vilkovisky algebra structure on the associated Chevalley-Eilenberg complex.

1. Introduction

The notion of Lie bialgebra was introduced by Drinfeld in [D1] in the context of the theory of Yang-Baxter equations. Later this notion played a fundamental role in his theory of Hopf algebra deformations of universal enveloping algebras, see the book [ES] and references cited therein.

Many interesting examples of Lie bialgebras automatically satisfy an additional algebraic condition, the so called *involutivity*, or “diamond” \diamond constraint. A remarkable example of such a Lie bialgebra structure was discovered by Turaev [T] on the vector space generated by all non-trivial free homotopy classes of curves on an orientable surface. Chas proved [Ch] that such a structure is in fact always involutive. This example was generalized to arbitrary manifolds within the framework of *string topology*: the equivariant homology of the free loop space of a compact manifold was shown by Chas and Sullivan [CS] to carry the structure of a graded involutive Lie bialgebra. An involutive Lie bialgebra structure was also found by Cieliebak and Latschev [CL] in the contact homology of an arbitrary exact symplectic manifold, while Schedler [S] introduced a natural involutive Lie bialgebra structure on the necklace Lie algebra associated to a quiver. It is worth pointing out that the construction of quantum A_∞ -algebras given in [B] (see also [H]) stems from the fact that the vector space of cyclic words in elements of a graded vector space W equipped with a (skew)symmetric pairing admits a canonical involutive Lie bialgebra structure. Therefore, involutive Lie bialgebras appear in many different areas of modern research.

In the study of the deformation theory of dg involutive Lie bialgebras one needs to know a minimal resolution of the associated properad. Such a minimal resolution is particularly nice and explicit if the properad happens to be *Koszul* [V1]. Koszulness of the prop(erad) of Lie bialgebras $\mathcal{Lie}\mathcal{B}$ was established by Markl and Voronov [MaVo] following an idea of Kontsevich [Ko]. The proof made use of a new category of *small props*, which are often called $\frac{1}{2}$ -*props* nowadays, and a new technical tool, the *path filtration* of a dg free properad. Attempts to settle the question of Koszulness or non-Koszulness of the properad of involutive Lie bialgebras, $\mathcal{Lie}\mathcal{B}^\diamond$, have been made since 2004. The Koszulness proof of $\mathcal{Lie}\mathcal{B}$ does not carry over to $\mathcal{Lie}\mathcal{B}^\diamond$ since the additional involutivity relation is not $\frac{1}{2}$ -properadic in nature. Motivated by some computer calculations the authors of [DCTT] conjectured in 2009 that the properad of involutive Lie bialgebras, $\mathcal{Lie}\mathcal{B}^\diamond$, is Koszul. In Section 2 of this paper we settle this long-standing problem. There are at least two not very straightforward steps in our solution. First, we extend Kontsevich’s exact functor from small props to props by twisting it with the relative simplicial cohomologies of graphs involved. This step allows us to incorporate operations in arities $(1, 1)$, $(1, 0)$ and $(0, 1)$ into the story, which were strictly prohibited in the Kontsevich construction as they fail exactness of his functor. Second, we reduce the cohomology computation of some important auxiliary

dg properad to a computation checking Koszulness of some ordinary quadratic algebra, which might be of independent interest.

By Koszul duality theory of properads [V1], our result implies immediately that the properad of non-unital Frobenius algebras is Koszul. By the curved Koszul duality theory [HM], the latter result implies, after some extra work, the Koszulness of the prop of unital-counital Frobenius algebras. These Frobenius properads also admit many applications in various areas of mathematics and mathematical physics, e. g. in representation theory, algebraic geometry, combinatorics, and recently, in 2-dimensional topological quantum field theory.

Another main result of this paper is a construction of a highly non-trivial action of the Grothendieck-Teichmüller group GRT_1 [D2] on minimal models of the operads of involutive Lie bialgebras/Frobenius algebras, and hence on the sets of homotopy involutive Lie bialgebra/Frobenius structures on an arbitrary dg vector space \mathfrak{g} . The Grothendieck-Teichmüller group GRT_1 has recently been shown to include a pro-unipotent subgroup freely generated by an infinite number of generators [Br], hence our construction provides a rich class of universal symmetries of the aforementioned objects.

In §5 of this paper we establish a surprising link between the theory of involutive Lie bialgebras and the operad of framed little disks whose cohomology was proven by Getzler [Ge] to be the operad \mathcal{BV} of Batalin-Vilkovisky algebras, an important structure in mathematical physics and algebra. We prove that, for any homotopy involutive Lie bialgebra structure on a complex \mathfrak{g} , there is an associated $\mathcal{BV}_\infty^{com}$ algebra (and hence a \mathcal{BV}_∞ algebra) structure on the associated Chevalley-Eilenberg complex $\odot^\bullet(\mathfrak{g}[-1])$. The operad $\mathcal{BV}_\infty^{com}$ (introduced in [Kr]) occurs naturally not only in our story, but also in the construction of the Koszul resolution of the operad \mathcal{BV} in [GTV], and in its recent applications to (Poisson) geometry. We furthermore prove that the cohomology of the dg operad $\mathcal{BV}_\infty^{com}$ is isomorphic to the operad \mathcal{BV} .

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Some notation. In this paper \mathbb{K} denotes a field of characteristic 0. The set $\{1, 2, \dots, n\}$ is abbreviated to $[n]$. Its group of automorphisms is denoted by \mathbb{S}_n . The sign representation of \mathbb{S}_n is denoted by sgn_n . The cardinality of a finite set A is denoted by $\#A$. If $V = \bigoplus_{i \in \mathbb{Z}} V^i$ is a graded vector space, then $V[k]$ stands for the graded vector space with $V[k]^i := V^{i+k}$. For $v \in V^i$ we set $|v| := i$. The phrase *differential graded* is abbreviated by dg. The n -fold symmetric product of a (dg) vector space V is denoted by $\odot^n V$, the full symmetric product space by $\odot^\bullet V$ or just $\odot V$ and the completed symmetric product by $\hat{\odot}^\bullet V$. For a finite group G acting on a vector space V , we denote via V^G the space of invariants with respect to the action of G , and by V_G the space of coinvariants $V_G = V/\{gv - v | v \in V, g \in G\}$. We always work over a field \mathbb{K} of characteristic zero so that, for finite G , we have a canonical isomorphism $V_G \cong V^G$.

We use freely the language of operads and properads and their Koszul duality theory. For a background on operads we refer to the textbook [LV], while the Koszul duality theory of properads has been developed in [V1]. For a properad \mathcal{P} we denote by $\mathcal{P}\{k\}$ the unique properad which has the following property: for any graded vector space V there is a one-to-one correspondence between representations of $\mathcal{P}\{k\}$ in V and representations of \mathcal{P} in $V[-k]$; in particular, $\mathcal{E}nd_V\{k\} = \mathcal{E}nd_V[-k]$. For \mathcal{C} a coaugmented co(pr)operad, we will denote by $\Omega(\mathcal{C})$ its cobar construction. Concretely, $\Omega(\mathcal{C}) = \mathcal{F}ree(\bar{\mathcal{C}}[1])$ as a graded (pr)operad where $\bar{\mathcal{C}}$ the cokernel of the coaugmetation and $\mathcal{F}ree(\dots)$ denotes the free (pr)operad generated by an \mathbb{S} -(bi)module. We will often complexes of derivations of (pr)operads and deformation complexes of (pr)operad maps. The notation $\text{Der}(\mathcal{P})$ shall be used for the complex of derivations of \mathcal{P} , considered as non-unital properad. For a map of properads $f : \Omega(\mathcal{C}) \rightarrow \mathcal{P}$, we will denote by

$$\text{Def}(\Omega(\mathcal{C}) \xrightarrow{f} \mathcal{P}) \cong \prod_{n,m} \text{Hom}_{\mathbb{S}_n \times \mathbb{S}_m}(\mathcal{C}(n, m), \mathcal{P}(n, m))$$

the associated convolution complex.

2. Koszulness of the prop of involutive Lie bialgebras

2.1. Involutive Lie bialgebras. A *Lie bialgebra* is a graded vector space \mathfrak{g} , equipped with degree zero linear maps,

$$\Delta: \mathfrak{g} \rightarrow \mathfrak{g} \wedge \mathfrak{g} \quad \text{and} \quad [\cdot, \cdot]: \wedge^2 \mathfrak{g} \rightarrow \mathfrak{g},$$

such that

- the data (\mathfrak{g}, Δ) is a Lie coalgebra;
- the data $(\mathfrak{g}, [\cdot, \cdot])$ is a Lie algebra;
- the compatibility condition,

$$\Delta [a, b] = \sum a_1 \otimes [a_2, b] + [a, b_1] \otimes b_2 + (-1)^{|a||b|} ([b, a_1] \otimes a_2 + b_1 \otimes [b_2, a]),$$

holds for any $a, b \in \mathfrak{g}$. Here $\Delta a =: \sum a_1 \otimes a_2$, $\Delta b =: \sum b_1 \otimes b_2$.

A Lie bialgebra $(\mathfrak{g}, [\cdot, \cdot], \Delta)$ is called *involutive* if the composition map

$$\begin{array}{ccccc} V & \xrightarrow{\Delta} & \Lambda^2 V & \xrightarrow{[\cdot, \cdot]} & V \\ a & \longrightarrow & \sum a' \otimes a'' & \longrightarrow & \sum [a', a''] \end{array}$$

vanishes. A dg (involutive) Lie bialgebra is a complex (\mathfrak{g}, d) equipped with the structure of an (involutive) Lie algebra such that the maps $[\cdot, \cdot]$ and Δ are morphisms of complexes.

2.1.1. An example. Let W be a finite dimensional graded vector space over a field \mathbb{K} of characteristic zero equipped with a degree 0 skewsymmetric pairing,

$$\begin{array}{ccc} \omega: W \otimes W & \longrightarrow & \mathbb{K} \\ w_1 \otimes w_2 & \longrightarrow & \omega(w_1, w_2) = -(-1)^{|w_1||w_2|} \omega(w_2, w_1). \end{array}$$

Then the associated vector space of “cyclic words in W ”,

$$Cyc^\bullet(W) := \bigoplus_{n \geq 0} (W^{\otimes n})_{\mathbb{Z}_n},$$

admits an involutive Lie bialgebra structure given by (see, e.g., [Go] and references cited there)

$$[(w_1 \otimes \dots \otimes w_n)_{\mathbb{Z}_n}, (v_1 \otimes \dots \otimes v_m)_{\mathbb{Z}_m}] := \sum_{\substack{i \in [n] \\ j \in [m]}} \pm \omega(w_i, v_j) (w_1 \otimes \dots \otimes w_{i-1} \otimes v_{j+1} \otimes \dots \otimes v_m \otimes v_1 \otimes \dots \otimes v_{j-1} \otimes w_{i+1} \otimes \dots \otimes w_n)_{\mathbb{Z}_{n+m-2}}$$

and

$$\Delta (w_1 \otimes \dots \otimes w_n)_{\mathbb{Z}_n} := \sum_{i \neq j} \pm \omega(w_i, w_j) (w_{i+1} \otimes \dots \otimes w_{j-1})_{\mathbb{Z}_{j-i-1}} \otimes (w_{j+1} \otimes \dots \otimes w_{i-1})_{\mathbb{Z}_{n-j+i-1}}$$

This example has many applications in various areas of modern research (see, e.g., [B, Ch, CL, H]).

2.2. Properad of involutive Lie bialgebras. By definition, the properad, $\mathcal{L}ie^\circ \mathcal{B}$, of involutive Lie bialgebras is a quadratic properad given as the quotient,

$$\mathcal{L}ie^\circ \mathcal{B} := Free(E) / \langle \mathcal{R} \rangle,$$

of the free properad generated by an \mathbb{S} -bimodule $E = \{E(m, n)\}_{m, n \geq 1}$ with all $E(m, n) = 0$ except

$$E(2, 1) := \mathbf{1}_1 \otimes sgn_2 = \text{span} \left\langle \begin{array}{c} 1 \\ \circlearrowleft \\ 1 \end{array} \begin{array}{c} 2 \\ \circlearrowright \\ 1 \end{array} = - \begin{array}{c} 2 \\ \circlearrowleft \\ 1 \end{array} \begin{array}{c} 1 \\ \circlearrowright \\ 1 \end{array} \right\rangle$$

$$E(1, 2) := sgn_2 \otimes \mathbf{1}_1 = \text{span} \left\langle \begin{array}{c} 1 \\ \circlearrowleft \\ 1 \end{array} \begin{array}{c} 1 \\ \circlearrowright \\ 2 \end{array} = - \begin{array}{c} 1 \\ \circlearrowleft \\ 2 \end{array} \begin{array}{c} 1 \\ \circlearrowright \\ 1 \end{array} \right\rangle$$

modulo the ideal generated by the following relations

$$(1) \quad \mathcal{R} : \left\{ \begin{array}{l} \begin{array}{c} 1 \quad 2 \\ \diagdown \quad / \\ \circ \\ / \quad \diagdown \\ 1 \quad 2 \end{array} + \begin{array}{c} 3 \quad 1 \\ \diagdown \quad / \\ \circ \\ / \quad \diagdown \\ 2 \quad 3 \end{array} + \begin{array}{c} 2 \quad 3 \\ \diagdown \quad / \\ \circ \\ / \quad \diagdown \\ 1 \quad 2 \end{array} = 0, \\ \begin{array}{c} 1 \\ | \\ \circ \\ / \quad \diagdown \\ 1 \quad 2 \end{array} + \begin{array}{c} 1 \\ | \\ \circ \\ / \quad \diagdown \\ 3 \quad 2 \end{array} + \begin{array}{c} 1 \\ | \\ \circ \\ / \quad \diagdown \\ 2 \quad 3 \end{array} = 0, \\ \begin{array}{c} 1 \quad 2 \\ \diagdown \quad / \\ \circ \\ / \quad \diagdown \\ 1 \quad 2 \end{array} - \begin{array}{c} 1 \quad 2 \\ \diagdown \quad / \\ \circ \\ / \quad \diagdown \\ 1 \quad 1 \end{array} + \begin{array}{c} 1 \quad 2 \\ \diagdown \quad / \\ \circ \\ / \quad \diagdown \\ 2 \quad 1 \end{array} - \begin{array}{c} 1 \quad 1 \\ \diagdown \quad / \\ \circ \\ / \quad \diagdown \\ 2 \quad 1 \end{array} + \begin{array}{c} 1 \quad 1 \\ \diagdown \quad / \\ \circ \\ / \quad \diagdown \\ 1 \quad 2 \end{array} = 0, \\ \begin{array}{c} \circ \\ / \quad \diagdown \\ \circ \\ / \quad \diagdown \\ \circ \end{array} = 0. \end{array} \right.$$

The properad governing Lie bialgebras $\mathcal{L}ie\mathcal{B}$ is defined in the same manner, except that the last relation of (1) is omitted.

Recall [V1] that any quadratic properad \mathcal{P} has an associated Koszul dual coproperad \mathcal{P}^i such that its cobar construction,

$$\Omega(\mathcal{P}^i) = \mathcal{F}ree\langle \bar{\mathcal{P}}^i[1] \rangle,$$

comes equipped with a differential d and with a canonical surjective map of dg properads,

$$(\Omega(\mathcal{P}^i), d) \longrightarrow (\mathcal{P}, 0).$$

This map always induces an isomorphism in cohomology in degree 0. If, additionally, the map is a quasi-isomorphism, then the properad \mathcal{P} is called *Koszul*. In this case the cobar construction $\Omega(\mathcal{P}^i)$ gives us a minimal resolution of \mathcal{P} and is denoted by \mathcal{P}_∞ . It is well known, for example, that the properad governing Lie bialgebras $\mathcal{L}ie\mathcal{B}$ is Koszul [MaVo].

We shall study below the Koszul dual properad $\mathcal{L}ie^\circ\mathcal{B}^i$, its cobar construction $\Omega(\mathcal{L}ie^\circ\mathcal{B}^i)$ and prove that the natural surjection $\Omega(\mathcal{L}ie^\circ\mathcal{B}^i) \rightarrow \mathcal{L}ie^\circ\mathcal{B}$ is a quasi-isomorphism. Anticipating this conclusion, we often use the symbol $\mathcal{L}ie^\circ\mathcal{B}_\infty$ as a shorthand for $\Omega(\mathcal{L}ie^\circ\mathcal{B}^i)$.

2.3. An explicit description of the dg properad $\mathcal{L}ie^\circ\mathcal{B}_\infty$. The Koszul dual of $\mathcal{L}ie^\circ\mathcal{B}$ is a coproperad $\mathcal{L}ie^\circ\mathcal{B}^i$ whose linear dual, $(\mathcal{L}ie^\circ\mathcal{B}^i)^*$, is the properad generated by degree 1 corollas,

$$\begin{array}{c} 1 \quad 2 \\ \diagdown \quad / \\ \bullet \\ / \quad \diagdown \\ 1 \end{array} = - \begin{array}{c} 2 \quad 1 \\ \diagdown \quad / \\ \bullet \\ / \quad \diagdown \\ 1 \end{array}, \quad \begin{array}{c} 1 \\ | \\ \bullet \\ / \quad \diagdown \\ 1 \quad 2 \end{array} = - \begin{array}{c} 1 \\ | \\ \bullet \\ / \quad \diagdown \\ 2 \quad 1 \end{array}$$

with the following relations,


$$\begin{array}{c} 1 \quad 2 \\ \diagdown \quad / \\ \bullet \\ / \quad \diagdown \\ 1 \quad 3 \end{array} = - \begin{array}{c} 2 \quad 3 \\ \diagdown \quad / \\ \bullet \\ / \quad \diagdown \\ 1 \quad 2 \end{array}, \quad \begin{array}{c} 1 \\ | \\ \bullet \\ / \quad \diagdown \\ 1 \quad 3 \end{array} = - \begin{array}{c} 1 \\ | \\ \bullet \\ / \quad \diagdown \\ 2 \quad 3 \end{array}, \quad \begin{array}{c} 1 \quad 2 \\ \diagdown \quad / \\ \bullet \\ / \quad \diagdown \\ 1 \quad 2 \end{array} = - \begin{array}{c} 1 \quad 2 \\ \diagdown \quad / \\ \bullet \\ / \quad \diagdown \\ 1 \quad 2 \end{array}.$$

Hence the following graphs

$$(2) \quad \begin{array}{c} \vdots \\ | \\ \vdots \\ | \\ \circ \\ / \quad \diagdown \\ \circ \\ / \quad \diagdown \\ \vdots \\ | \\ \vdots \end{array}$$

where

$$(3) \quad \begin{array}{c} \circ \\ | \\ \vdots \\ | \\ \circ \end{array} := \begin{array}{c} \diamond \\ | \\ \vdots \\ | \\ \diamond \end{array}$$

is the composition of $a = 0, 1, 2, \dots$ graphs of the form , form a basis of $(\mathcal{L}ie^\circ\mathcal{B}^i)^*$. If the graph (2) has n input legs and m output legs, then it has $m+n+2a-2$ vertices and its degree is equal to $m+n+2a-2$. Hence the properad $\Omega((\mathcal{L}ie^\circ\mathcal{B}^i) = \mathcal{F}ree\langle(\mathcal{L}ie^\circ\mathcal{B}^i)[1]\rangle$ is a free properad generated by the following skewsymmetric corollas of degree $3-m-n-2a$,

$$(4) \quad \begin{array}{c} 1 \quad 2 \quad \dots \quad m \\ \diagdown \quad \diagup \\ \textcircled{a} \\ \diagup \quad \diagdown \\ 1 \quad 2 \quad \dots \quad n \end{array} = (-1)^{\sigma+\tau} \begin{array}{c} \sigma(1) \sigma(2) \dots \sigma(m) \\ \diagdown \quad \diagup \\ \textcircled{a} \\ \diagup \quad \diagdown \\ \tau(1) \tau(2) \dots \tau(n) \end{array} \quad \forall \sigma \in \mathbb{S}_m, \forall \tau \in \mathbb{S}_n,$$

where $m+n+a \geq 3$, $m \geq 1$, $n \geq 1$, $a \geq 0$. The non-negative number a is called the *weight* of the generating corolla (4). The differential in $\Omega((\mathcal{L}ie^\circ\mathcal{B}^i)$ is given by¹

$$(5) \quad \delta \begin{array}{c} 1 \quad 2 \quad \dots \quad m \\ \diagdown \quad \diagup \\ \textcircled{a} \\ \diagup \quad \diagdown \\ 1 \quad 2 \quad \dots \quad n \end{array} = \sum_{a=b+c+l-1} \sum_{\substack{[m]=I_1 \sqcup I_2 \\ [n]=J_1 \sqcup J_2}} \pm \begin{array}{c} \textcircled{c} \\ \diagup \quad \diagdown \\ \textcircled{b} \\ \diagdown \quad \diagup \\ \textcircled{a} \\ \diagup \quad \diagdown \\ J_1 \quad J_2 \end{array}$$

where the parameter l counts the number of internal edges connecting the two vertices on the right-hand side. We have, in particular,

$$\delta \begin{array}{c} | \\ \textcircled{0} \\ | \end{array} = 0, \quad \delta \begin{array}{c} \diagdown \quad \diagup \\ \textcircled{0} \\ \diagup \quad \diagdown \end{array} = 0, \quad \delta \begin{array}{c} | \\ \textcircled{1} \\ | \end{array} = \begin{array}{c} \textcircled{0} \\ | \\ \textcircled{0} \\ | \end{array}$$

so that the map

$$(6) \quad \pi : \mathcal{L}ie^\circ\mathcal{B}_\infty \longrightarrow \mathcal{L}ie^\circ\mathcal{B},$$

which sends to zero all generators of $\mathcal{L}ie^\circ\mathcal{B}_\infty$ except the following ones,

$$\pi \left(\begin{array}{c} | \\ \textcircled{0} \\ | \end{array} \right) = \begin{array}{c} | \\ \wedge \\ | \end{array}, \quad \pi \left(\begin{array}{c} \diagdown \quad \diagup \\ \textcircled{0} \\ \diagup \quad \diagdown \end{array} \right) = \begin{array}{c} \diagdown \quad \diagup \\ \vee \\ \diagdown \quad \diagup \end{array},$$

is a morphism of dg properads, as expected. To prove that the properad $\mathcal{L}ie^\circ\mathcal{B}$ is Koszul is equivalent to showing that the map π is a quasi-isomorphism. As $\mathcal{L}ie^\circ\mathcal{B}_\infty$ is non-negatively graded, the map π is a quasi-isomorphism if and only if the cohomology of the dg properad $\mathcal{L}ie^\circ\mathcal{B}_\infty$ is concentrated in degree zero. We shall prove this property below in §2.8 with the help of several auxiliary constructions which we discuss next.

2.4. A decomposition of the complex $\mathcal{L}ie^\circ\mathcal{B}_\infty$. As a vector space the properad $\mathcal{L}ie^\circ\mathcal{B}_\infty = \{\mathcal{L}ie^\circ\mathcal{B}_\infty(m, n)\}_{m, n \geq 1}$ is spanned by oriented graphs built from corollas (4). For such a graph $\Gamma \in \mathcal{L}ie^\circ\mathcal{B}_\infty$ we set

$$\|\Gamma\| := g(\Gamma) + w(\Gamma) \in \mathbb{N}$$

where $g(\Gamma)$ is its genus and $w(\Gamma)$ is its total weight defined as the sum of weights of its vertices(-corollas). It is obvious that the differential δ in $\mathcal{L}ie^\circ\mathcal{B}_\infty$ respects this total grading,

$$\|\delta\Gamma\| = \|\Gamma\|.$$

Therefore each complex $(\mathcal{L}ie^\circ\mathcal{B}_\infty(m, n), \delta)$ decomposes into a direct sum of subcomplexes,

$$\mathcal{L}ie^\circ\mathcal{B}_\infty(m, n) = \sum_{s \geq 0} \mathcal{L}ie^\circ\mathcal{B}_\infty(m, n)^{(s)}$$

where

$$\mathcal{L}ie^\circ\mathcal{B}_\infty(m, n)^{(s)} := \text{span}\langle\{\Gamma : \text{graphs } \Gamma \in \mathcal{L}ie^\circ\mathcal{B}_\infty(m, n) \text{ with } \|\Gamma\| = s\}\rangle$$

¹The precise sign factors in this formula can be determined via a usual trick: the analogous differential in the degree shifted properad $\Omega((\mathcal{L}ie^\circ\mathcal{B}^i)\{1\})$ must be given by the same formula but with all sign factors equal to +1.

by

$$F(s)(m, n) = \bigoplus_{\Gamma \in \overline{\text{Gr}}(m, n)} \left(\bigotimes_{v \in v(\Gamma)} s(\text{Out}(v), \text{In}(v)) \otimes \odot H^1(\Gamma, \partial\Gamma) \right)_{\text{Aut}(\Gamma)},$$

where $\overline{\text{Gr}}(m, n)$ represents the set of all (isomorphism classes of) directed acyclic graphs with n output legs and m input legs that are irreducible in the sense that they do not allow any $\frac{1}{2}$ -prop-ic contractions. The differential acts trivially on the $H^1(\Gamma, \partial\Gamma)$ part.

2.6.2. Lemma. *F is an exact functor.*

Proof. Since the differential preserves the underlying graph, we get

$$(11) \quad H_{\bullet}(F(s)(m, n)) = \bigoplus_{\Gamma \in \overline{\text{Gr}}(m, n)} H_{\bullet} \left(\bigotimes_{v \in v(\Gamma)} s(\text{Out}(v), \text{In}(v)) \otimes \odot H^1(\Gamma, \partial\Gamma) \right)_{\text{Aut}(\Gamma)}$$

Since the differential commutes with elements of $\text{Aut}(\Gamma)$, $\text{Aut}(\Gamma)$ is finite and \mathbb{K} is a field of characteristic zero, by Maschke's Theorem we have

$$(12) \quad (11) = \bigoplus_{\Gamma \in \overline{\text{Gr}}(m, n)} \left(H_{\bullet} \left(\bigotimes_{v \in v(\Gamma)} s(\text{Out}(v), \text{In}(v)) \otimes \odot H^1(\Gamma, \partial\Gamma) \right) \right)_{\text{Aut}(\Gamma)}$$

Applying the Künneth formula twice, together with the fact that the differential is trivial on $H^1(\Gamma, \partial\Gamma)$ we get

$$(12) = \bigoplus_{\Gamma \in \overline{\text{Gr}}(m, n)} \left(\bigotimes_{v \in v(\Gamma)} H_{\bullet}(s(\text{Out}(v), \text{In}(v))) \otimes \odot H^1(\Gamma, \partial\Gamma) \right)_{\text{Aut}(\Gamma)} = F(H_{\bullet}(s))(m, n).$$

□

Step 2: A genus filtration.

Consider the genus filtration of $(\mathcal{L}ie^{\circ}\mathcal{B}_{\infty}, \delta)$, and denote by $(\text{gr}\mathcal{L}ie^{\circ}\mathcal{B}_{\infty}, \delta^{gen})$ the associated graded properad. The differential δ^{gen} in the complex $\text{gr}\mathcal{L}ie^{\circ}\mathcal{B}_{\infty}$ is given by the formula,

$$(13) \quad \delta^{gen} \begin{array}{c} 1 \quad 2 \quad \dots \quad m \\ \diagup \quad \diagdown \\ \odot \\ \diagdown \quad \diagup \\ 1 \quad 2 \quad \dots \quad n \end{array} = \sum_{a=b+c} \sum_{\substack{[m]=I_1 \sqcup I_2 \\ [n]=J_1 \sqcup J_2}} \pm \begin{array}{c} J_2 \\ \diagup \quad \diagdown \\ \odot \\ \diagdown \quad \diagup \\ J_1 \end{array}$$

Consider also the genus filtration of the dg properad \mathcal{P} and denote by $(\text{gr}\mathcal{L}ie^{\circ}\mathcal{B}_{\infty}\mathcal{P}, 0)$ the associated graded. The morphism (9) of filtered complexes induces a sequence of morphisms of the associated graded complexes,

$$(14) \quad \nu : \text{gr}\mathcal{L}ie^{\circ}\mathcal{B}_{\infty} \longrightarrow \text{gr}\mathcal{P},$$

Thanks to the Spectral Sequences Comparison Theorem, Proposition 2.6.1 will be proven if we show that the map ν is a quasi-isomorphism of complexes. We shall compute below the cohomology $H^{\bullet}(\text{gr}\mathcal{L}ie^{\circ}\mathcal{B}_{\infty}, \delta^{gen})$ which will make it evident that the map ν is an isomorphism indeed.

Step 3: An auxiliary prop. Let us consider a properad $\mathcal{Q} = F(\Omega_{\frac{1}{2}}(\mathcal{L}ie\mathcal{B}_{\frac{1}{2}}^i))$, where $\mathcal{L}ie\mathcal{B}_{\frac{1}{2}}^i$ is the $\frac{1}{2}$ -prop governing Lie bialgebras. Explicitly, \mathcal{Q} is generated by corollas with either $a = m = n = 1$ or $a = 0$ and $m + n \geq 3$ subject to the relations

$$\begin{array}{c} \textcircled{1} \\ | \\ \textcircled{1} \end{array} = 0, \quad \sum_{i=1}^m \begin{array}{c} i \\ \diagup \quad \diagdown \\ \textcircled{1} \\ \diagdown \quad \diagup \\ 1 \quad \dots \quad i-1 \quad i+1 \quad \dots \quad m \end{array} - \sum_{i=1}^n \begin{array}{c} 1 \quad 2 \quad \dots \quad m-1 \quad m \\ \diagup \quad \diagdown \\ \textcircled{0} \\ \diagdown \quad \diagup \\ 1 \quad \dots \quad i-1 \quad i+1 \quad \dots \quad n \\ | \\ \textcircled{1} \end{array} = 0$$

With this description we see that ν factors through \mathcal{Q} , $\nu : \text{grLie}^\diamond \mathcal{B}_\infty \xrightarrow{p} \mathcal{Q} \xrightarrow{q} \text{gr}\mathcal{P}$. Furthermore we claim that q is a quasi-isomorphism. First notice that $\text{gr}\mathcal{P} = F(\text{Lie}\mathcal{B}_{\frac{1}{2}})$. The $\frac{1}{2}$ -prop $\text{Lie}\mathcal{B}_{\frac{1}{2}}$ is Koszul, i. e., the natural projection $\Omega_{\frac{1}{2}}(\text{Lie}\mathcal{B}_{\frac{1}{2}}^i) \rightarrow \text{Lie}\mathcal{B}_{\frac{1}{2}}$ is a quasi-isomorphism. The result follows by applying the functor F to this map and by Lemma 2.6.2.

Step 4: p is a quasi-isomorphism.

Consider a filtration of $\text{grLie}^\diamond \mathcal{B}_\infty$ given by the sum of all decorations of non-bivalent vertices. On the 0-th page of this spectral sequence the differential acts only by splitting bivalent vertices. Then Proposition 2.5.1 tells us that the first page of this spectral sequence consists of graphs with no bivalent vertices such that every vertex is decorated by a number $a \in \mathbb{Z}^+$ and every edge has either a decoration $\textcircled{1}$ or no decoration. The differential acts by

$$(15) \quad \begin{array}{c} \textcircled{a} \\ \begin{array}{c} 1 \quad 2 \quad \dots \quad m \\ \diagup \quad \diagdown \\ \dots \\ \diagdown \quad \diagup \\ 1 \quad 2 \quad \dots \quad n \end{array} \end{array} \longrightarrow \sum_{i=1}^m \begin{array}{c} i \\ \textcircled{1} \\ \begin{array}{c} 1 \quad i-1 \quad i+1 \quad m \\ \diagup \quad \diagdown \\ \dots \\ \diagdown \quad \diagup \\ 1 \quad 2 \quad \dots \quad n-1 \quad n \end{array} \end{array} - \sum_{i=1}^n \begin{array}{c} 1 \quad 2 \quad \dots \quad m-1 \quad m \\ \diagup \quad \diagdown \\ \dots \\ \diagdown \quad \diagup \\ 1 \quad i-1 \quad \textcircled{1} \quad i+1 \quad n \end{array}$$

The complex we obtain is precisely

$$\bigoplus_{\Gamma \in \overline{\text{Gr}}(m,n)} \left(\bigotimes_{v \in v(\Gamma)} \Omega_{\frac{1}{2}}(\text{Lie}\mathcal{B}_{\frac{1}{2}}^i)(\text{Out}(v), \text{In}(v)) \otimes \odot C^*(\Gamma, \partial\Gamma) \right)_{\text{Aut}(\Gamma)},$$

where $C^*(\Gamma, \partial\Gamma)$ are the simplicial co-chains of Γ relative to its boundary; the differential in this complex is given by the standard differential in $C^*(\Gamma, \partial\Gamma)$. Indeed, we may identify $C^0(\Gamma, \partial\Gamma) \cong \mathbb{K}[V(\Gamma)]$ and $C^1(\Gamma, \partial\Gamma) \cong \mathbb{K}[E(\Gamma)]$. A vertex $v = \textcircled{a_v}$ with weight a_v corresponds to the a_v -th power of the cochain representing the vertex, and an edge decorated with the symbol $\textcircled{1}$ corresponds to the cochain representing the edge. The differential d on $C^1(\Gamma, \partial\Gamma)$ is the map dual to the standard boundary map $\partial : C_1(\Gamma, \partial\Gamma) \cong \mathbb{K}[E(\Gamma)] \rightarrow C_0(\Gamma, \partial\Gamma) \cong \mathbb{K}[V(\Gamma)]$. It is given, on a vertex $v \in V(\Gamma)$, by

$$dv = \sum_{e'_v \in \text{Out}(v)} e'_v - \sum_{e''_v \in \text{In}(v)} e''_v,$$

where $\text{Out}(v)$ is the set of edges outgoing from v and $\text{In}(v)$ is the set of edges ingoing to v . This exactly matches the differential (15) on the first page of the spectral sequence. As $H^0(\Gamma, \partial\Gamma) = 0$ and since the symmetric product functor \odot is exact we obtain \mathcal{Q} on the second page of the spectral sequence,

$$H^\bullet(\text{grLie}^\diamond \mathcal{B}_\infty) \cong \bigoplus_{\Gamma \in \overline{\text{Gr}}(m,n)} \left(\bigotimes_{v \in v(\Gamma)} \Omega_{\frac{1}{2}}(\text{Lie}\mathcal{B}_{\frac{1}{2}}^i)(\text{Out}(v), \text{In}(v)) \otimes \odot H^1(\Gamma, \partial\Gamma) \right)_{\text{Aut}(\Gamma)} = F(\Omega_{\frac{1}{2}}(\text{Lie}\mathcal{B}_{\frac{1}{2}}^i)) \cong \mathcal{Q}$$

thus showing that p is a quasi-isomorphism. \square

2.7. Auxiliary complexes. Let \mathcal{A}_n be a quadratic algebra generated by x_1, \dots, x_n with relations $x_i x_{i+1} = x_{i+1} x_i$ for $i = 1, \dots, n-1$. We denote by $C_n = \mathcal{A}_n^*$ the Koszul dual coalgebra. Notice that \mathcal{A}_n and C_n are weight graded and the weight k component of C_n , $C_n^{(k)}$ is zero if $k \geq 3$, while $C_n^{(1)} = \text{span}\{x_1, \dots, x_n\}$ and $C_n^{(2)} = \text{span}\{u_{1,2} = x_1 x_2 - x_2 x_1, u_{2,3} = x_2 x_3 - x_3 x_2, \dots, u_{n-1,n} = x_{n-1} x_n - x_n x_{n-1}\}$.

2.7.1. Proposition. *The algebra \mathcal{A}_n is Koszul. In particular, the canonical projection map $A_n := \Omega(C_n) \rightarrow \mathcal{A}_n$ from the cobar construction of C_n is a quasi-isomorphism.*

The proof of this proposition is given in Appendix A.

Proposition 2.7.1 in particular implies that the homology of the A_n vanishes in positive degree. The complex A_n is naturally multigraded by the amount of times each index j appears on each word and the differential respects this multigrading. We will be interested in particular in the subcomplex $\mathcal{A}_n^{1,1,\dots,1}$ of A_n that is spanned by words in x_j and $u_{i,i+1}$ such that each index occurs exactly once. Since $A_n^{1,1,\dots,1}$ is a direct summand of A_n , its homology also vanishes in positive degree.

Let us define a Lie algebra $\mathcal{L}_n = \text{Lie}(x_1, \dots, x_n)/[x_i, x_{i+1}]$ and a complex $L_n = \text{Lie}(x_1, \dots, x_n, u_{1,2}, \dots, u_{n-1,n})$, with $dx_i = 0$ and $d(u_{i,i+1}) = x_i x_{i+1} - x_{i+1} x_i$. Here Lie stands for the free Lie algebra functor.

2.7.2. Lemma. *The projection map $L_n \twoheadrightarrow \mathcal{L}_n$ is a quasi-isomorphism.*

Proof. It is clear that $H^0(L_n) = \mathcal{L}_n$, therefore it is enough to see that the homology of L_n vanishes in positive degree. The Poincaré-Birkhoff-Witt Theorem gives us an isomorphism

$$\odot(\text{Lie}(x_1, \dots, x_n, u_{1,2}, \dots, u_{n-1,n})) = \odot(L_n) \xrightarrow{\sim} \mathcal{A}ss(x_1, \dots, x_n, u_{1,2}, \dots, u_{n-1,n}) = A_n.$$

This map commutes with with the differentials, therefore we have an isomorphism in homology $H_\bullet(\odot(L_n)) = H_\bullet(A_n)$. Since \odot is an exact functor it commutes with taking homology and since the homology of A_n vanishes in positive degree by 2.7.1 the result follows. \square

Let us define A_{n_1, \dots, n_r} as the coproduct of A_{n_1}, \dots, A_{n_r} in the category of associative algebras; A_{n_1, \dots, n_r} consists of words in $x_1^1, x_2^1, \dots, x_{n_1}^1, x_1^2, \dots, x_{n_2}^2, \dots, x_1^r, \dots, x_{n_r}^r, u_{1,2}^1, \dots, u_{n_1-1, n_1}^1, u_{1,2}^2, \dots, u_{n_r-1, n_r}^r$. We define similarly L_{n_1, \dots, n_r} and $\mathcal{L}_{n_1, \dots, n_r}$.

2.7.3. Lemma. *The homology of A_{n_1, \dots, n_r} vanishes in positive degree.*

Proof. We prove the statement for $r = 2$ and the result follows by induction.

We can write $A_{n_1, n_2} = \bigoplus_\epsilon A_\epsilon$ where $\epsilon = (e_1, \dots, e_{|\epsilon|})$ represents a string of alternating n_1 's and n_2 's of finite size $|\epsilon|$. The differential preserves this decomposition. A_ϵ is naturally $|\epsilon|$ -multigraded.

Let $x = \sum \alpha_I x_I \in A_\epsilon$, where x_I represents some word in $x_i^1, x_i^2, u_{i,i+1}^1, u_{i,i+1}^2$ be a cycle of nonzero degree. Then, when we apply the Leibniz rule on the word x_I each parcel has a different multigrading, therefore the equation $dx = 0$ splits into $|\epsilon|$ equations whose right hand side is zero. For each one of them we can use the fact that the homology of A_{n_1} and A_{n_2} vanishes in non-zero degree to conclude that the left hand side of each of these equations is actually a boundary. It follows that x is a boundary as well. \square

2.7.4. Lemma. *The map $L_{n_1, \dots, n_r} \twoheadrightarrow \mathcal{L}_{n_1, \dots, n_r}$ is a quasi-isomorphism.*

Proof. The same argument from Lemma 2.7.2 holds. \square

2.7.5. Corollary. *The map $L_{n_1, \dots, n_r}^{1, \dots, 1} \twoheadrightarrow \mathcal{L}_{n_1, \dots, n_r}^{1, \dots, 1}$ is a quasi-isomorphism.*

2.8. Main Theorem. *The properad $\text{Lie}^\circ \mathcal{B}$ is Koszul, i.e. the natural surjection (6) a quasi-isomorphism.*

Proof. The surjection (6) factors through the surjection (9),

$$\pi : \text{Lie}^\circ \mathcal{B}_\infty \xrightarrow{\nu} \mathcal{P} \xrightarrow{\rho} \text{Lie}^\circ \mathcal{B}.$$

In view of Proposition 2.6.1, the Main theorem is proven once it is shown that the morphism ρ is a quasi-isomorphism. The latter statement is, in turn, proven once it is shown that the cohomology of the non-positively graded dg properad \mathcal{P} is concentrated in degree zero. For notation reasons it is suitable to work with the dg prop, \mathcal{PP} , generated by the properad \mathcal{P} . We also denote by $\text{Lie}P$ the prop governing Lie algebras and by $\text{Lie}CP$ the prop governing Lie coalgebras.

It is easy to see that the dg prop $\mathcal{PP} = \{\mathcal{PP}(m, n)\}$ is isomorphic, as a dg \mathbb{S} -bimodule, to the dg prop generated by a degree 1 corolla \bullet , degree zero corollas $\begin{array}{c} 1 \quad 2 \\ \diagdown \quad / \\ \circ \end{array} = - \begin{array}{c} 2 \quad 1 \\ \diagdown \quad / \\ \circ \end{array}$ and $\begin{array}{c} 1 \\ \diagdown \quad / \\ \circ \quad 2 \end{array} = - \begin{array}{c} 2 \\ \diagdown \quad / \\ \circ \quad 1 \end{array}$, modulo the first three relations of (1) and the following ones,

$$(16) \quad \begin{array}{c} \bullet \\ | \\ \bullet \end{array} = 0, \quad \begin{array}{c} \bullet \\ | \\ \circ \\ / \quad \backslash \\ \bullet \quad \bullet \end{array} = 0, \quad \begin{array}{c} \circ \\ / \quad \backslash \\ \bullet \quad \bullet \end{array} = 0.$$

The latter complex can in turn be identified with the following collection of dg vector spaces,

$$(17) \quad W(n, m) := \bigoplus_N (\text{Lie}P(n, N) \otimes V^{\otimes N} \otimes \text{Lie}CP(N, m))_{\mathbb{S}_N}$$

where V is a two-dimensional vector space $V_0 \oplus V_1$, where $V_0 = \text{span}\langle \left| \right\rangle$ and $V_1 = \text{span}\langle \left| \bullet \right\rangle$ and the differential in this complex is given by formulae (8).

Let us consider a slightly different complex

$$(18) \quad \begin{aligned} V_{n,m} &= \bigoplus_N (\mathcal{L}ieP(n, N) \otimes V^{\otimes N} \otimes \mathcal{A}ssCP(N, m))_{\mathbb{S}_N} \\ &\cong \bigoplus_N \bigoplus_{N=n_1+\dots+n_m} (\mathcal{L}ieP(n, N) \otimes V^{\otimes N} \otimes \mathcal{A}ssC(n_1) \otimes \dots \otimes \mathcal{A}ssC(n_m))_{\mathbb{S}_{n_1} \times \dots \times \mathbb{S}_{n_m}}, \end{aligned}$$

where $\mathcal{A}ssCP$ is the prop governing coassociative coalgebras.

The operad $\mathcal{A}ss = \mathcal{C}om \circ \mathcal{L}ie = \bigoplus_k (\mathcal{C}om \circ \mathcal{L}ie)^{(k)}$ is naturally graded with respect to the arity in $\mathcal{C}om$. This decomposition induces a multigrading in $V_{n,m} = \bigoplus_{(k_1, \dots, k_m)} V_{n; k_1, \dots, k_m}$. It is clear that this decomposition is

actually a splitting of complexes and in fact the direct summand $V_{1, \dots, 1}$ is just $W(n, m)$, therefore to show the theorem it suffices to show that the cohomology of $V_{n,m}$ is zero in positive degree.

There is a natural identification $\mathcal{L}ieP(n, N) \cong \left((\mathcal{L}ie(y_1, \dots, y_N))^{\otimes n} \right)^{1, \dots, 1}$ where, as before, we use the notation $1, \dots, 1$ to represent the subspace spanned by words such that each index appears exactly once. $(\mathcal{L}ieP(n, N) \otimes \mathcal{A}ssC(n_1) \otimes \dots \otimes \mathcal{A}ssC(n_m))_{\mathbb{S}_{n_1} \times \dots \times \mathbb{S}_{n_m}}$ is isomorphic to $\mathcal{L}ieP(n, N)$ but there is a more natural identification than the one above, namely the y_j can be gathered by blocks of size n_j , according to the action of $S_1 \times \dots \times S_{n_m}$ on $\mathcal{L}ieP(n, N)$ and can be relabeled accordingly:

$$y_1, y_2, \dots, y_N \rightsquigarrow y_1^1, \dots, y_{n_1}^1, y_1^2, \dots, y_{n_2}^2, \dots, y_1^m, \dots, y_{n_m}^m.$$

Since $V = V_0 \oplus V_1$, there is a natural decomposition of $V^{\otimes N} = \bigoplus_{\epsilon} V_{\epsilon_1} \otimes \dots \otimes V_{\epsilon_N}$, where $\epsilon = (\epsilon_1, \dots, \epsilon_N)$ runs through all strings of 0's and 1 of length N . Then,

$$\begin{aligned} (\mathcal{L}ieP(n, N) \otimes V^{\otimes N} \otimes \mathcal{A}ssC(n_1) \otimes \dots \otimes \mathcal{A}ssC(n_m))_{\mathbb{S}_{n_1} \times \dots \times \mathbb{S}_{n_m}} &= \\ \bigoplus_{\epsilon} (\mathcal{L}ieP(n, N) \otimes V_{\epsilon_1} \otimes \dots \otimes V_{\epsilon_N} \otimes \mathcal{A}ssC(n_1) \otimes \dots \otimes \mathcal{A}ssC(n_m))_{\mathbb{S}_{n_1} \times \dots \times \mathbb{S}_{n_m}} & \end{aligned}$$

Given a fixed a string ϵ we look at each summand individually. If ϵ_k is zero we wish to correspond V_{ϵ_k} to an element of the form x_j^i and if $\epsilon_k = 1$ we wish to correspond V_{ϵ_k} to an element of the form $u_{j,j+1}^i$. We do this recursively, to determine the correct indices:

- For $k = 1$: $\begin{cases} \text{if } \epsilon_1 = 0, \text{ then } V_{\epsilon_1} = \text{span} \langle x_1^1 \rangle; \\ \text{if } \epsilon_1 = 1 \text{ then } V_{\epsilon_1} = \text{span} \langle u_{1,2}^1 \rangle. \end{cases}$
- For $k > 1$:
 - if $\epsilon_k = 0$ then
 - if $V_{\epsilon_{k-1}} = \text{span} \langle x_j^i \rangle$ or $V_{\epsilon_{k-1}} = \text{span} \langle u_{j-1,j}^i \rangle$ and the total number of variables with upper script i is still smaller than n_i , then $V_{\epsilon_k} = \text{span} \langle x_{j+1}^i \rangle$.
 - Otherwise $V_{\epsilon_k} = \text{span} \langle x_1^{i+1} \rangle$.
 - if $\epsilon_k = 1$ then
 - if $V_{\epsilon_{k-1}} = \text{span} \langle x_j^i \rangle$ or $V_{\epsilon_{k-1}} = \text{span} \langle u_{j-1,j}^i \rangle$ and the total number of variables with upper script i is still smaller than n_i , then $V_{\epsilon_k} = \text{span} \langle u_{j+1,j+2}^i \rangle$.
 - Otherwise $V_{\epsilon_k} = \text{span} \langle u_{1,2}^{i+1} \rangle$.

With this notation, the total space

$$V_{n,m} = \bigoplus_N \bigoplus_{\epsilon} \bigoplus_{N=n_1+\dots+n_m} (\mathcal{L}ieP(n, N) \otimes V_{\epsilon_1} \otimes \dots \otimes V_{\epsilon_N} \otimes \mathcal{A}ssC(n_1) \otimes \dots \otimes \mathcal{A}ssC(n_m))_{\mathbb{S}_{n_1} \times \dots \times \mathbb{S}_{n_m}}$$

2.9), and in particular $\mathcal{L}ie\mathcal{B}^i \cong (\mathcal{F}rob_2^{\circ})^* \{1\}$. It then follows from the Koszulness of $\mathcal{L}ie\mathcal{B}$ (and its degree shifted relatives) that the properads $\mathcal{F}rob_a^{\circ}$ are Koszul, as noted in [JF].

The properad $uc\mathcal{F}rob$ of unital-counital Frobenius algebras is, by definition, a quotient of the free properad generated by degree zero corollas \circlearrowleft (unit), \circlearrowright (counit), $\begin{smallmatrix} 1 & 2 \\ \diagdown & \diagup \\ \circ & \\ \diagup & \diagdown \\ 2 & 1 \end{smallmatrix}$ (graded co-commutative comultiplication) and $\begin{smallmatrix} 1 & 2 \\ \diagdown & \diagup \\ \circ & \\ \diagup & \diagdown \\ 1 & 2 \end{smallmatrix}$ (graded commutative multiplication) modulo the ideal generated by the relations (19) and the additional relations

$$(20) \quad \begin{smallmatrix} \circ & \\ \diagdown & \diagup \\ \circ & \end{smallmatrix} - \begin{smallmatrix} | \\ \circ \end{smallmatrix} = 0 \quad , \quad \begin{smallmatrix} \circ & \\ \diagup & \diagdown \\ \circ & \end{smallmatrix} - \begin{smallmatrix} | \\ \circ \end{smallmatrix} = 0 \quad .$$

where the vertical line $\begin{smallmatrix} | \\ \circ \end{smallmatrix}$ stands for the unit in the properad $\mathcal{F}rob$. Similarly one defines a properad $u\mathcal{F}rob$ of unital Frobenius algebras, and a properad $c\mathcal{F}rob$ of counital algebras. Clearly, $u\mathcal{F}rob$ and $c\mathcal{F}rob$ are subproperads of $uc\mathcal{F}rob$.

2.10.2. Theorem. *The properads $u\mathcal{F}rob$, $c\mathcal{F}rob$ and $uc\mathcal{F}rob$ are Koszul.*

Proof. By curved Koszul duality theory [HM], it is enough to prove Koszulness of the associated quadratic properads, $qu\mathcal{F}rob$, $qc\mathcal{F}rob$ and $quc\mathcal{F}rob$, obtained from $u\mathcal{F}rob$, $c\mathcal{F}rob$ and, respectively, $uc\mathcal{F}rob$ by replacing inhomogeneous relations (20) by the following ones (see §3.6 in [HM]),

$$\begin{smallmatrix} \circ & \\ \diagdown & \diagup \\ \circ & \end{smallmatrix} = 0 \quad , \quad \begin{smallmatrix} \circ & \\ \diagup & \diagdown \\ \circ & \end{smallmatrix} = 0$$

so that we have decompositions into direct sums of \mathbb{S} -bimodules,

$$qu\mathcal{F}rob = \text{span} \left\langle \begin{smallmatrix} \text{coCom} \\ \circlearrowleft \\ \circ \end{smallmatrix} \right\rangle \oplus \mathcal{F}rob \quad , \quad qc\mathcal{F}rob = \text{span} \left\langle \begin{smallmatrix} \circlearrowright \\ \circ \\ \text{Com} \end{smallmatrix} \right\rangle \oplus \mathcal{F}rob$$

and

$$quc\mathcal{F}rob = \text{span} \left\langle \begin{smallmatrix} \text{coCom} \\ \circlearrowleft \\ \circ \end{smallmatrix} \right\rangle \oplus \text{span} \left\langle \begin{smallmatrix} \circlearrowright \\ \circ \\ \text{Com} \end{smallmatrix} \right\rangle \oplus \mathcal{F}rob.$$

where $\begin{smallmatrix} \circlearrowleft \\ \circ \end{smallmatrix}$ stands for the graph given in (3).

Consider, for example, the properad $qc\mathcal{F}rob$ (proofs of Koszulness of properads $qu\mathcal{F}rob$ and $quc\mathcal{F}rob$ can be given by a similar argument). Its Koszul dual properad $qc\mathcal{F}rob^! =: qc\mathcal{L}ie^{\circ}\mathcal{B}$ is generated by the properads $\mathcal{L}ie^{\circ}\mathcal{B}$ and $\begin{smallmatrix} \circlearrowright \\ \circ \end{smallmatrix}$ modulo the following relation,

$$\begin{smallmatrix} \circlearrowright \\ \circ \end{smallmatrix} = 0.$$

We have,

$$qc\mathcal{L}ie^{\circ}\mathcal{B}^i = (qc\mathcal{F}rob)^* \{1\} \cong \text{span} \left\langle \begin{smallmatrix} \circlearrowright \\ \circ \\ \mathcal{L}ie^i \end{smallmatrix} \right\rangle \oplus \mathcal{L}ie^{\circ}\mathcal{B}^i.$$

It will suffice to show that the properad $qc\mathcal{L}ie^{\circ}\mathcal{B}$ is Koszul. To this end consider the dg properad $\Omega(qc\mathcal{L}ie^{\circ}\mathcal{B}^i)$ which is a free properad generated by corollas (4) and the following ones,

$$(21) \quad \begin{smallmatrix} \circlearrowright \\ \circ \\ 1 \dots n \end{smallmatrix} = (-1)^{\sigma} \begin{smallmatrix} \circlearrowright \\ \circ \\ \sigma(1) \dots \sigma(n) \end{smallmatrix}$$

where $a \geq 0$, $n \geq 1$ and $\sigma \in \mathbb{S}_n$ is an arbitrary permutation. The differential is given on corollas (4) by the standard formula (5) and on $(0, n)$ -generators by

$$d \begin{smallmatrix} \circlearrowright \\ \circ \\ 1 \dots n \end{smallmatrix} = \sum_{a=b+c+l-1} \sum_{\substack{[n]=J_1 \sqcup J_2 \\ \#J_1 \geq 1}} \pm \begin{smallmatrix} \circlearrowright \\ \circ \\ J_1 \dots J_2 \end{smallmatrix}$$

where l counts the number of internal edges connecting the two vertices on the right-hand side. There is a natural morphism of properads

$$\Omega(qc\mathcal{L}ie^\circ\mathcal{B}^i) \longrightarrow qc\mathcal{L}ie^\circ\mathcal{B},$$

which is a quasi-isomorphism if and only if $qc\mathcal{L}ie^\circ\mathcal{B}$ is Koszul. Thus to prove Koszulness of $qc\mathcal{L}ie^\circ\mathcal{B}$ it is enough to establish an isomorphism $H^\bullet(\Omega(qc\mathcal{L}ie^\circ\mathcal{B}^i)) \cong qc\mathcal{L}ie^\circ\mathcal{B}$ of \mathbb{S} -bimodules.

To do this, one may closely follow the proof of Theorem 2.8, adjusting it slightly so as to allow for the additional $(0, n)$ -ary generators. First, we define a properad $\tilde{\mathcal{P}}$ which is generated by the properad \mathcal{P} of section 2.6, together with an additional generator of arity $(0, 1)$, in pictures $\overset{\circ}{\downarrow}$, with the additional relations

$$\begin{array}{c} \circ \\ \diagup \quad \diagdown \\ \text{---} \end{array} = 0 \qquad \qquad \qquad \overset{\circ}{\downarrow} = 0.$$

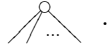
The map $\Omega(qc\mathcal{L}ie^\circ\mathcal{B}^i) \rightarrow qc\mathcal{L}ie^\circ\mathcal{B}$ clearly factors through $\tilde{\mathcal{P}}$

$$(22) \qquad \qquad \qquad \Omega(qc\mathcal{L}ie^\circ\mathcal{B}^i) \rightarrow \tilde{\mathcal{P}} \rightarrow qc\mathcal{L}ie^\circ\mathcal{B}$$

and it suffices to show that both of the above maps are quasi-isomorphisms. Consider first the left-hand map. The fact that this map is a quasi-isomorphism may be proven by copying the proof of Theorem 2.6.1, except that now the functor F (as in section 2.6) is applied not to the cobar construction, $\Omega_{\frac{1}{2}}(\mathcal{L}ie\mathcal{B}_{\frac{1}{2}}^i)$ but to $\Omega_{\frac{1}{2}}(qc\mathcal{L}ie\mathcal{B}_{\frac{1}{2}}^i)$. Here

$$qc\mathcal{L}ie\mathcal{B}_{\frac{1}{2}}^i(m, n) = \begin{cases} \mathcal{L}ie\mathcal{B}_{\frac{1}{2}}^i(m, n) & \text{if } (m, n) \neq (0, 1) \\ \mathbb{K} & \text{if } (m, n) = (0, 1) \end{cases}$$

is the $\frac{1}{2}$ -prop governing Lie bialgebras with a counit operation killed by the cobracket. More concretely, $qc\mathcal{L}ie\mathcal{B}_{\frac{1}{2}}^i(m, n)$ is the same as $\mathcal{L}ie\mathcal{B}_{\frac{1}{2}}^i(m, n)$ in all arities (m, n) with $m, n > 0$, but $qc\mathcal{L}ie\mathcal{B}_{\frac{1}{2}}^i(0, n)$ is one-dimensional, the extra operations corresponding to corollas



One can check² that the $\frac{1}{2}$ -prop $qc\mathcal{L}ie\mathcal{B}_{\frac{1}{2}}^i$ is Koszul, i. e., that

$$H(\Omega_{\frac{1}{2}}(qc\mathcal{L}ie\mathcal{B}_{\frac{1}{2}}^i)) \cong qc\mathcal{L}ie\mathcal{B}_{\frac{1}{2}}^i.$$

The properad $\tilde{\mathcal{P}}$ is obtained by applying the exact functor F to this $\frac{1}{2}$ -prop, and hence, by essentially the same arguments as in the proof of Theorem 9 the left-hand map of (22) is a quasi-isomorphism.

Next consider the right hand map of (22). It can be shown to be a quasi-isomorphism along the lines of the proof of Theorem 2.8. Again, it is clear that the degree zero cohomology of $\tilde{\mathcal{P}}$ is $qc\mathcal{L}ie^\circ\mathcal{B}$, so it will suffice to show that $H^{>0}(\tilde{\mathcal{P}}) = 0$. First, let $\widetilde{\mathcal{P}\mathcal{P}}$ be the prop generated by the properad $\tilde{\mathcal{P}}$. As a dg \mathbb{S} -bimodule it is isomorphic to (cf. (17))

$$\tilde{W}(n, m) := \bigoplus_{N, M} (\mathcal{L}ieP(n, N) \otimes V^{\otimes N} \otimes \mathbb{K}^{\otimes M} \otimes \mathcal{L}ieCP(N + M, m))_{\mathbb{S}_N \times \mathbb{S}_M}$$

where V is as in (17). The above complex $\tilde{W}(n, m)$ is a direct summand of the complex (cf. (18))

$$\tilde{V}_{n, m} := \bigoplus_{N, M} (\mathcal{L}ieP(n, N) \otimes V^{\otimes N} \otimes \mathbb{K}^{\otimes M} \otimes \mathcal{A}ssCP(N + M, m))_{\mathbb{S}_N \times \mathbb{S}_M}$$

by arguments similar to those following (18). Then again by the Koszulness results of section 2.7 it follows that the above complex has no cohomology in positive degrees, hence neither can $\widetilde{\mathcal{P}\mathcal{P}}$ have cohomology in positive degrees. Hence we can conclude that the properad $qc\mathcal{L}ie^\circ\mathcal{B}$ is Koszul. \square

²The piece of the $\frac{1}{2}$ -prop $\Omega_{\frac{1}{2}}(qc\mathcal{L}ie\mathcal{B}_{\frac{1}{2}}^i)$ involving the additional generators is isomorphic to the complex $(E_1^{\text{Lie}+}, d_1^{\text{Lie}+})$ from [Me2] (see page 344). According to loc. cit. its cohomology is one-dimensional.

3. Deformation complexes

As one application of the Koszulness of $\mathcal{L}ie^\circ\mathcal{B}$ and $\mathcal{F}rob$ we obtain minimal models $\mathcal{L}ie^\circ\mathcal{B}_\infty = \Omega(\mathcal{L}ie^\circ\mathcal{B}^i) = \Omega(\mathcal{F}rob^*\{1\})$ and $\mathcal{F}rob_\infty = \Omega((\mathcal{L}ie^\circ\mathcal{B})^*\{1\})$ of these properads and hence minimal models for their deformation complexes and for the deformation complexes of their algebras.

3.1. A deformation complex of an involutive Lie bialgebra. According to the general theory [MeVa], $\mathcal{L}ie^\circ\mathcal{B}_\infty$ -algebra structures in a dg vector space (\mathfrak{g}, d) can be identified with Maurer-Cartan elements,

$$\mathcal{MC}(\text{InvLieB}(\mathfrak{g})) := \{\Gamma \in \text{InvLieB}(\mathfrak{g}) : |\Gamma| = 3 \text{ and } [\Gamma, \Gamma]_{CE} = 0\},$$

of a graded Lie algebra,

$$(23) \quad \text{InvLieB}(\mathfrak{g}) := \text{Def} \left(\mathcal{L}ie^\circ\mathcal{B}_\infty \xrightarrow{0} \mathcal{E}nd_{\mathfrak{g}} \right) [-2],$$

which controls deformations of the zero morphism from $\mathcal{L}ie^\circ\mathcal{B}_\infty$ to the endomorphism properad $\mathcal{E}nd_{\mathfrak{g}} = \{\text{Hom}(\mathfrak{g}^{\otimes n}, \mathfrak{g}^{\otimes m})\}$. As a \mathbb{Z} -graded vector space $\text{InvLieB}(\mathfrak{g})$ can be identified with the vector space of homomorphisms of \mathbb{S} -bimodules,

$$\begin{aligned} \text{InvLieB}(\mathfrak{g}) &= \text{Hom}_{\mathbb{S}}((\mathcal{L}ie^\circ\mathcal{B})^i, \mathcal{E}nd_{\mathfrak{g}}) [-2] \\ &= \prod_{\substack{a \geq 0, m, n \geq 1 \\ m+n+a \geq 3}} \text{Hom}_{\mathbb{S}_m \times \mathbb{S}_n}(\text{sgn}_n \otimes \text{sgn}_m[m+n+2a-2], \text{Hom}(\mathfrak{g}^{\otimes n}, \mathfrak{g}^{\otimes m})) [-2] \\ &= \prod_{\substack{a \geq 0, m, n \geq 1 \\ m+n+a \geq 3}} \text{Hom}(\odot^n(\mathfrak{g}[-1]), \odot^m(\mathfrak{g}[-1]))[-2a] \\ &\subset \widehat{\odot}(\mathfrak{g}[-1] \oplus \mathfrak{g}^*[-1] \oplus \mathbb{K}[-2]) \simeq \mathbb{K}[[\eta^i, \psi_i, \hbar]] \end{aligned}$$

where \hbar a formal parameter of degree 2 (a basis vector of the summand $\mathbb{K}[-2]$ above), and, for a basis $(e_1, e_2, \dots, e_i, \dots)$ in \mathfrak{g} and the associated dual basis $(e^1, e^2, \dots, e^i, \dots)$ in \mathfrak{g}^* we set $\eta^i := s e^i$, $\psi_i := s e_i$, where $s : V \rightarrow V[-1]$ is the suspension map. Therefore the Lie algebra $\text{InvLieB}(\mathfrak{g})$ has a canonical structure of a module over the algebra $\mathbb{K}[[\hbar]]$; moreover, for finite dimensional \mathfrak{g} its elements can be identified with formal power series³, f , in variables ψ_i, η^i and \hbar , which satisfy the “boundary” conditions,

$$(24) \quad f(\psi, \eta, \hbar)|_{\psi_i=0} = 0, \quad f(\psi, \eta, \hbar)|_{\eta^i=0} = 0, \quad f(\psi, \eta, \hbar)|_{\hbar=0} \in I^3$$

where I is the maximal ideal in $\mathbb{K}[[\psi, \eta]]$. The Lie brackets in $\text{InvLieB}(\mathfrak{g})$ can be read off either from the coproperad structure in $(\mathcal{L}ie^\circ\mathcal{B})^i$ or directly from the formula (5) for the differential, and are given explicitly by (cf. [DCTT]),

$$(25) \quad [f, g]_{\hbar} := f *_\hbar g - (-1)^{|f||g|} g *_\hbar f$$

where (up to Koszul signs),

$$f *_\hbar g := \sum_{k=0}^{\infty} \frac{\hbar^{k-1}}{k!} \sum_{i_1, \dots, i_k} \pm \frac{\partial^k f}{\partial \eta^{i_1} \dots \partial \eta^{i_k}} \frac{\partial^k g}{\partial \psi_{i_1} \dots \partial \psi_{i_k}}$$

is an associative product. Note that the differential $d_{\mathfrak{g}}$ in \mathfrak{g} gives rise to a quadratic element, $D_{\mathfrak{g}} = \sum_{i,j} \pm d_j^i \psi_i \eta^j$, of homological degree 3 in $\mathbb{K}[[\eta^i, \psi_i, \hbar]]$, where d_j^i are the structure constants of $d_{\mathfrak{g}}$ in the chosen basis, $d_{\mathfrak{g}}(e_i) =: \sum_j d_j^i e_j$.

Finally, we can identify $\mathcal{L}ie^\circ\mathcal{B}_\infty$ structures in a finite dimensional dg vector space $(\mathfrak{g}, d_{\mathfrak{g}})$ with a homogenous formal power series,

$$\Gamma := D_{\mathfrak{g}} + f \in \mathbb{K}[[\eta^i, \psi_i, \hbar]],$$

³In fact, this is true for a class of infinite-dimensional vector spaces. Consider a category of graded vector spaces which are inverse limits of finite dimensional ones (with the corresponding topology and with the completed tensor product), and also a category of graded vector spaces which are direct limits of finite dimensional ones. If \mathfrak{g} belongs to one of these categories, then \mathfrak{g}^* belongs (almost by definition) to the other, and we have isomorphisms of the type $(\mathfrak{g} \otimes \mathfrak{g})^* = \mathfrak{g}^* \otimes \mathfrak{g}^*$ which are required for the “local coordinate” formulae to work.

of homological degree 3 such that

$$(26) \quad \Gamma \star_{\hbar} \Gamma = \sum_{k=0}^{\infty} \frac{\hbar^{k-1}}{k!} \sum_{i_1, \dots, i_k} \pm \frac{\partial^k \Gamma}{\partial \eta^{i_1} \dots \partial \eta^{i_k}} \frac{\partial^k \Gamma}{\partial \psi_{i_1} \dots \partial \psi_{i_k}} = 0,$$

and the summand f satisfies boundary conditions (24).

For example, let

$$(\Delta : V \rightarrow \wedge^2 V, \quad [,] : \wedge^2 V \rightarrow V)$$

be a Lie bialgebra structure in a vector space V which we assume for simplicity to be concentrated in degree 0. Let C_{ij}^k and Φ_k^{ij} be the associated structure constants,

$$[x_i, x_j] =: \sum_{k \in I} C_{ij}^k x_k, \quad \Delta(x_k) =: \sum_{i, j \in I} \Phi_k^{ij} x_i \wedge x_j.$$

Then it is easy to check that all the involutive Lie bialgebra axioms (1) get encoded into a single equation $\Gamma \star_{\hbar} \Gamma = 0$ for $\Gamma := \sum_{i, j, k \in I} (C_{ij}^k \psi_k \eta^i \eta^j + \Phi_k^{ij} \eta^k \psi_i \psi_j)$.

Note that all the above formulae taken modulo the ideal generated by the formal variable \hbar give us a Lie algebra,

$$(27) \quad \text{LieB}(\mathfrak{g}) := \text{Def} \left(\mathcal{L}ie\mathcal{B}_{\infty} \xrightarrow{0} \mathcal{E}nd_{\mathfrak{g}} \right) [-2] \cong \mathbb{K}[[\psi_i, \eta^i]]$$

controlling the deformation theory of (not-necessarily involutive) Lie bialgebra structures in a dg space \mathfrak{g} . Lie brackets in (27) are given in coordinates by the standard Poisson formula,

$$(28) \quad \{f, g\} = \sum_{i \in I} (-1)^{|f||\eta^i|} \frac{\partial f}{\partial \psi_i} \frac{\partial g}{\partial \eta^i} - (-1)^{|f||\psi_i|} \frac{\partial f}{\partial \eta^i} \frac{\partial g}{\partial \psi_i}$$

for any $f, g \in \mathbb{K}[[\psi_i, \eta^i]]$. Formal power series, $f \in \mathbb{K}[[\psi_i, \eta^i]]$, which have homological degree 3 and satisfy the equations,

$$\{f, f\} = 0, \quad f(\psi, \eta)|_{\psi_i=0} = 0, \quad f(\psi, \eta)|_{\eta^i=0} = 0,$$

are in one-to-one correspondence with strongly homotopy Lie bialgebra structures in a finite dimensional dg vector space \mathfrak{g} .

3.2. Deformation complexes of properads. The deformation complex of a properad \mathcal{P} is by definition the dg Lie algebra $\text{Der}(\tilde{\mathcal{P}})$ of derivations of a cofibrant resolution $\tilde{\mathcal{P}} \xrightarrow{\sim} \mathcal{P}$. It may be identified as a complex with the deformation complex of the identity map $\tilde{\mathcal{P}} \rightarrow \tilde{\mathcal{P}}$ (which controls deformations of \mathcal{P} -algebras) up to a degree shift:

$$\text{Der}(\tilde{\mathcal{P}}) \cong \text{Def}(\tilde{\mathcal{P}} \rightarrow \tilde{\mathcal{P}})[1].$$

Note however that both $\text{Der}(\mathcal{P})$ and $\text{Def}(\tilde{\mathcal{P}} \rightarrow \tilde{\mathcal{P}})$ have natural dg Lie (or Lie_{∞}) algebra structures that are *not* preserved by the above map. Furthermore, there is a quasi-isomorphism of dg Lie algebras

$$(29) \quad \text{Def}(\tilde{\mathcal{P}} \rightarrow \tilde{\mathcal{P}}) \rightarrow \text{Def}(\tilde{\mathcal{P}} \rightarrow \mathcal{P})$$

The zeroth cohomology $H^0(\text{Der}(\tilde{\mathcal{P}}))$ is of particular importance. It is a differential graded Lie algebra whose elements act on the space of $\tilde{\mathcal{P}}$ algebra structures on any vector space. We shall see that in the examples we are interested in this dg Lie algebra is very rich, and that it acts non-trivially in general.

Using the Koszulness of the properads $\mathcal{L}ie\mathcal{B}$, $\mathcal{F}rob^{\diamond}$ from [MaVo, Ko] and the Koszulness of $\mathcal{L}ie^{\circ}\mathcal{B}$ and $\mathcal{F}rob$ from Theorem 2.8 and Corollary 2.10.1 we can write down the following models for the deformation

$$\delta\Gamma = \delta_{\mathcal{L}ie\mathcal{B}_\infty}\Gamma \pm \sum \text{graph}_1 \pm \sum \text{graph}_2$$

FIGURE 1. A graph interpretation of an element of $\text{Der}(\mathcal{L}ie\mathcal{B}_\infty)$, and the pictorial description of the differential.

$$\delta\Gamma = \delta_{\mathcal{L}ie^\circ\mathcal{B}_\infty}\Gamma \pm \sum \hbar^{p+k-1} \text{graph}_1 \pm \sum \hbar^{p+k-1} \text{graph}_2$$

FIGURE 2. A graph interpretation of an element of $\text{Der}(\mathcal{L}ie^\circ\mathcal{B}_\infty)$, and the pictorial description of the differential.

complexes.

$$\begin{aligned} \text{Der}(\mathcal{L}ie\mathcal{B}_\infty) &= \prod_{n,m \geq 1} \text{Hom}_{\mathbb{S}_n \times \mathbb{S}_m}((\mathcal{F}rob^\circ)^*\{1\}(n,m), \mathcal{L}ie\mathcal{B}_\infty(n,m))[1] \\ &\cong \prod_{n,m \geq 1} (\mathcal{L}ie\mathcal{B}_\infty(n,m) \otimes \text{sgn}_n \otimes \text{sgn}_m)^{\mathbb{S}_n \times \mathbb{S}_m}[-n-m+1] \\ \text{Der}(\mathcal{L}ie^\circ\mathcal{B}_\infty) &= \prod_{n,m \geq 1} \text{Hom}_{\mathbb{S}_n \times \mathbb{S}_m}(\mathcal{F}rob^*\{1\}(n,m), \mathcal{L}ie^\circ\mathcal{B}_\infty(n,m))[1] \\ &\cong \prod_{n,m \geq 1} (\mathcal{L}ie^\circ\mathcal{B}_\infty(n,m) \otimes \text{sgn}_n \otimes \text{sgn}_m)^{\mathbb{S}_n \times \mathbb{S}_m}[[\hbar]][-n-m+1] \\ \text{Der}(\mathcal{F}rob_\infty) &= \prod_{n,m \geq 1} \text{Hom}_{\mathbb{S}_n \times \mathbb{S}_m \geq 1}((\mathcal{L}ie^\circ\mathcal{B})^*\{1\}(n,m), \mathcal{F}rob_\infty(n,m))[1] \\ \text{Der}(\mathcal{F}rob_\infty^\circ) &= \prod_{n,m \geq 1} \text{Hom}_{\mathbb{S}_n \times \mathbb{S}_m \geq 1}(\mathcal{L}ie\mathcal{B}^*\{1\}(n,m), \mathcal{F}rob_\infty^\circ(n,m))[1] \end{aligned}$$

Here \hbar is a formal variable of degree 2. Each of the models on the right has a natural combinatorial interpretation as a graph complex, cf. also [MaVo, section 1.7]. For example $\text{Der}(\mathcal{L}ie\mathcal{B}_\infty)$ may be interpreted as a complex of directed graphs which have incoming and outgoing legs but have no closed paths of directed edges. The differential is obtained by splitting vertices and by attaching new vertices at one of the external legs, see Figure 1.

Similarly, $\text{Der}(\mathcal{L}ie^\circ\mathcal{B}_\infty)$ may be interpreted as a complex of \hbar -power series of graphs with weighted vertices. The differential is obtained by splitting vertices and attaching vertices at external legs as indicated in Figure 2. The Lie bracket is combinatorially obtained by inserting graphs into vertices of another. We leave it to the reader to work out the structure of the graph complexes and the differentials for the complexes $\text{Der}(\mathcal{F}rob_\infty)$ and $\text{Der}(\mathcal{F}rob_\infty^\circ)$.

The cohomology of all these graph complexes is hard to compute. We may however simplify the computation by using formula (29) and equivalently compute instead

$$\begin{aligned}
\text{Def}(\mathcal{L}ie\mathcal{B}_\infty \rightarrow \mathcal{L}ie\mathcal{B}) &= \prod_{n,m} \text{Hom}_{\mathbb{S}_n \times \mathbb{S}_m}((\mathcal{F}rob^\diamond)^*\{1\}(n,m), \mathcal{L}ie\mathcal{B}(n,m)) \\
&\cong \prod_{n,m} (\mathcal{L}ie\mathcal{B}(n,m) \otimes \text{sgn}_n \otimes \text{sgn}_m)^{\mathbb{S}_n \times \mathbb{S}_m}[-n-m] \\
\text{Def}(\mathcal{L}ie^\diamond\mathcal{B}_\infty \rightarrow \mathcal{L}ie^\diamond\mathcal{B}) &= \prod_{n,m} \text{Hom}_{\mathbb{S}_n \times \mathbb{S}_m}(\mathcal{F}rob^*\{1\}(n,m), \mathcal{L}ie^\diamond\mathcal{B}(n,m)) \\
&\cong \prod_{n,m} (\mathcal{L}ie^\diamond\mathcal{B}(n,m) \otimes \text{sgn}_n \otimes \text{sgn}_m)^{\mathbb{S}_n \times \mathbb{S}_m}[-n-m][[\hbar]] \\
\text{Def}(\mathcal{F}rob_\infty \rightarrow \mathcal{F}rob) &= \prod_{n,m} \text{Hom}_{\mathbb{S}_m \times \mathbb{S}_n}((\mathcal{L}ie^\diamond\mathcal{B})^*\{1\}(n,m), \mathcal{F}rob(n,m)) \\
\text{Def}(\mathcal{F}rob_\infty^\diamond \rightarrow \mathcal{F}rob^\diamond) &= \prod_{n,m} \text{Hom}_{\mathbb{S}_n \times \mathbb{S}_m \geq 1}(\mathcal{L}ie\mathcal{B}^*\{1\}(n,m), \mathcal{F}rob^\diamond(n,m)).
\end{aligned}$$

Note however that here we lose the dg Lie algebra structure, or rather there is a different Lie algebra structure on the above complexes. The above complexes may again be interpreted as graph complexes. For example $\text{Def}(\mathcal{L}ie\mathcal{B}_\infty \rightarrow \mathcal{L}ie\mathcal{B})$ consists of directed trivalent acyclic graphs with incoming and outgoing legs, modulo the Jacobi and Drinfeld five term relations. The differential is obtained by attaching a trivalent vertex at one external leg in all possible ways.

Finally we note that of the above four deformation complexes only two are essentially different. For example, note that $\text{Hom}_{\mathbb{S}_n \times \mathbb{S}_m}(\mathcal{L}ie\mathcal{B}^*\{1\}(n,m), \mathcal{F}rob^\diamond(n,m))$ is just a completion of

$$\text{Hom}_{\mathbb{S}_n \times \mathbb{S}_m}((\mathcal{F}rob^\diamond)^*\{1\}(n,m), \mathcal{L}ie\mathcal{B}(n,m)) \cong (\mathcal{F}rob^\diamond(n,m) \otimes \text{sgn}_n \otimes \text{sgn}_m) \otimes_{\mathbb{S}_n \times \mathbb{S}_m} \mathcal{L}ie\mathcal{B}(n,m)[-n+m]$$

Concretely, the completion is with respect to the genus grading of $\mathcal{L}ie\mathcal{B}$, and the differential preserves the genus grading. Hence the cohomology of one complex is just the completion of the cohomology of the other with respect to the genus grading.

Similar arguments show that the cohomologies $\text{Def}(\mathcal{L}ie^\diamond\mathcal{B}_\infty \rightarrow \mathcal{L}ie^\diamond\mathcal{B})$ and $\text{Def}(\mathcal{F}rob_\infty \rightarrow \mathcal{F}rob)$ are the same up to completion issues. Here the differential does not preserve the genus but preserves the quantity (genus)-(\hbar -degree). Hence it suffices to discuss one of each pair of deformation complexes. We will discuss $\text{Def}(\mathcal{L}ie\mathcal{B}_\infty \rightarrow \mathcal{L}ie\mathcal{B})$ and $\text{Def}(\mathcal{L}ie^\diamond\mathcal{B}_\infty \rightarrow \mathcal{L}ie^\diamond\mathcal{B})$ in the next section.

4. Oriented graph complexes and the universal \mathfrak{gt}_1 action

The goal of this section is to reduce the computation of the above deformation complexes to the computation of the cohomology of M. Kontsevich's graph complex. By a result of one of the authors [W1] the degree zero cohomology of this graph complex agrees with the Grothendieck-Teichmüller Lie algebra \mathfrak{gt}_1 . This will allow us to conclude that the Grothendieck-Teichmüller group universally acts on $\mathcal{F}rob_\infty$ and $\mathcal{L}ie^\diamond\mathcal{B}_\infty$ structures. This extends the well known result that the Grothendieck-Teichmüller group universally acts on Lie bialgebra structures.

4.1. Grothendieck-Teichmüller group. The profinite and pronipotent Grothendieck-Teichmüller groups were introduced by Vladimir Drinfeld in his study of braid groups and quasi-Hopf algebras. They turned out to be one of the most interesting and mysterious objects in modern mathematics. The profinite Grothendieck-Teichmüller group \widehat{GT} plays an important role in number theory and algebraic geometry. The pro-unipotent Grothendieck-Teichmüller group GT (and its graded version GRT) over a field of characteristic zero was used by Pavel Etingof and David Kazhdan to solve the Drinfeld's quantization conjecture for Lie bialgebras. Maxim Kontsevich's and Dmitry Tamarkin's formality theory unravels the role of the group GRT in the deformation quantization of Poisson structures. Later Anton Alekseev and Charles Torossian applied GRT to the Kashiwara-Vergne problem in Lie theory. The Grothendieck-Teichmüller group unifies different fields, and every time this group appears in a mathematical theory, there follows a breakthrough in that theory. We refer to Hidekazu Furusho's lecture note [F] for precise definitions and references.

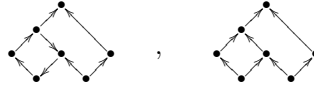
In this paper we consider the Grothendieck-Teichmüller group GRT_1 which is the kernel of the canonical morphism of groups $GRT \rightarrow \mathbb{K}^*$. As GRT_1 is pronounipotent, it is of the form $\exp(\mathfrak{grt}_1)$ for some Lie algebra \mathfrak{grt}_1 whose definition can be found, for example, in §6 of [W1]. Therefore to understand representations of GRT_1 is the same as to understand representations of the Grothendieck-Teichmüller Lie algebra \mathfrak{grt}_1 .

4.2. Completed versions of $\mathcal{L}ie^\circ\mathcal{B}$ and $\mathcal{L}ie\mathcal{B}$. The properads $\mathcal{L}ie\mathcal{B}$ and $\mathcal{L}ie^\circ\mathcal{B}$ are naturally graded by the genus of the graphs describing the operations. We will denote by $\widehat{\mathcal{L}ie^\circ\mathcal{B}}$ and $\widehat{\mathcal{L}ie\mathcal{B}}$ the completions with respect to this grading. Similarly, we denote by $\widehat{\mathcal{L}ie\mathcal{B}}_\infty$ the completion of $\mathcal{L}ie\mathcal{B}_\infty$ with respect to the genus grading. The natural map $\widehat{\mathcal{L}ie\mathcal{B}}_\infty \rightarrow \widehat{\mathcal{L}ie\mathcal{B}}$ is a quasi-isomorphism. Furthermore, we denote by $\widehat{\mathcal{L}ie^\circ\mathcal{B}}_\infty$ the completion of $\mathcal{L}ie^\circ\mathcal{B}_\infty$ with respect to the genus plus the total weight-grading, i. e., with respect to the grading $\|\cdot\|$ described in section 2.4. Then the map $\widehat{\mathcal{L}ie^\circ\mathcal{B}}_\infty \rightarrow \widehat{\mathcal{L}ie^\circ\mathcal{B}}$ is a quasi-isomorphism.

We will call a continuous representation of $\widehat{\mathcal{L}ie\mathcal{B}}$ (respectively of $\widehat{\mathcal{L}ie^\circ\mathcal{B}}$) a *genus complete* (involutive) Lie bialgebra. Here the topology on $\widehat{\mathcal{L}ie\mathcal{B}}$ (respectively on $\widehat{\mathcal{L}ie^\circ\mathcal{B}}$) is the one induced by the genus filtration (respectively the filtration $\|\cdot\|$). For example, the involutive Lie bialgebra discussed in section 2.1.1 is clearly genus complete since both the cobracket and the bracket reduce the lengths of the cyclic words.

Abusing notation slightly we will denote by $\text{Der}(\widehat{\mathcal{L}ie\mathcal{B}}_\infty)$ (respectively by $\text{Der}(\widehat{\mathcal{L}ie^\circ\mathcal{B}}_\infty)$) the complex of *continuous* derivations. The sub-properads $\mathcal{L}ie\mathcal{B}_\infty \subset \widehat{\mathcal{L}ie\mathcal{B}}_\infty$ and $\mathcal{L}ie^\circ\mathcal{B}_\infty \subset \widehat{\mathcal{L}ie^\circ\mathcal{B}}_\infty$ are dense by definition and hence any continuous derivation is determined by its restriction to these sub-properads. It also follows that the above complexes of derivations are isomorphic as complexes to $\text{Def}(\mathcal{L}ie\mathcal{B}_\infty \rightarrow \widehat{\mathcal{L}ie\mathcal{B}}_\infty)[1]$ and $\text{Def}(\mathcal{L}ie^\circ\mathcal{B}_\infty \rightarrow \widehat{\mathcal{L}ie^\circ\mathcal{B}}_\infty)[1]$. Finally we note that the cohomology of these complexes is merely the completion of the cohomology of the complexes $\text{Der}(\mathcal{L}ie\mathcal{B}_\infty)$ and $\text{Der}(\mathcal{L}ie^\circ\mathcal{B}_\infty)$, since the differential respects the gradings.

4.3. An operad of graphs $\mathcal{G}ra^\uparrow$. A graph is called *directed* if its edges are equipped with directions as in the following examples,



A directed graph is called *oriented* or *acyclic* if it contains no directed *closed* paths of edges. For example, the second graph above is oriented while the first one is not. For arbitrary integers $n \geq 1$ and $l \geq 0$ let $\mathbf{G}_{n,l}^\uparrow$ stand for the set of oriented graphs, $\{\Gamma\}$, with n vertices and l edges such that the vertices of Γ are labelled by elements of $[n] := \{1, \dots, n\}$, i.e. an isomorphism $V(\Gamma) \rightarrow [n]$ is fixed.

Let $\mathbb{K}\langle\mathbf{G}_{n,l}^\uparrow\rangle$ be the vector space over a field \mathbb{K} of characteristic zero which is spanned by graphs from $\mathbf{G}_{n,l}^\uparrow$, and consider a \mathbb{Z} -graded \mathbb{S}_n -module,

$$\mathcal{G}ra^\uparrow(n) := \bigoplus_{l=0}^{\infty} \mathbb{K}\langle\mathbf{G}_{n,l}^\uparrow\rangle[2l].$$

For example, $\overset{1}{\bullet} \longrightarrow \overset{2}{\bullet}$ is a degree -2 element in $\mathcal{G}ra^\uparrow(2)$. The \mathbb{S} -module, $\mathcal{G}ra^\uparrow := \{\mathcal{G}ra(n)^\uparrow\}_{n \geq 1}$, is naturally an operad with the operadic compositions given by

$$(30) \quad \begin{array}{ccc} \circ_i : \mathcal{G}ra^\uparrow(n) \otimes \mathcal{G}ra^\uparrow(m) & \longrightarrow & \mathcal{G}ra^\uparrow(m+n-1) \\ \Gamma_1 \otimes \Gamma_2 & \longrightarrow & \sum_{\Gamma \in \mathbf{G}_{m+n-1}^i} \Gamma \end{array}$$

where \mathbf{G}_{m+n-1}^i is the subset of $\mathbf{G}_{m+n-1, \#E(\Gamma_1) + \#E(\Gamma_2)}^\uparrow$ consisting of graphs, Γ , satisfying the condition: the full subgraph of Γ spanned by the vertices labeled by the set $\{i, i+1, \dots, i+m-1\}$ is isomorphic to Γ_2 and the quotient graph Γ/Γ_2 (which is obtained from Γ obtained by contracting that subgraph Γ_2 to a single vertex) is isomorphic to Γ_1 , see, e.g., §7 in [Me1] or §2 in [W1] for explicit examples of this kind of operadic compositions. The unique element in $\mathbf{G}_{1,0}^\uparrow$ serves as the unit in the operad $\mathcal{G}ra^\uparrow$.

4.3.1. A representation of $\mathcal{G}ra^\uparrow$ in $\text{LieB}(\mathfrak{g})$. For any graded vector space \mathfrak{g} the operad $\mathcal{G}ra^\uparrow$ has a natural representation in the associated graded vector space $\text{LieB}(\mathfrak{g})$ (see (27)),

$$(31) \quad \begin{array}{ccc} \rho : \mathcal{G}ra^\uparrow(n) & \longrightarrow & \text{End}_{\text{LieB}(\mathfrak{g})}(n) = \text{Hom}(\text{LieB}(\mathfrak{g})^{\otimes n}, \text{LieB}(\mathfrak{g})) \\ \Gamma & \longrightarrow & \Phi_\Gamma \end{array}$$

given by the formula,

$$\begin{array}{ccc} \Phi_\Gamma : \otimes^n \text{LieB}(\mathfrak{g}) & \longrightarrow & \text{LieB}(\mathfrak{g}) \\ \gamma_1 \otimes \dots \otimes \gamma_n & \longrightarrow & \Phi_\Gamma(\gamma_1, \dots, \gamma_n) := \mu \left(\prod_{e \in E(\Gamma)} \Delta_e(\gamma_1(\psi) \otimes \gamma_2(\psi) \otimes \dots \otimes \gamma_n(\psi)) \right) \end{array}$$

where, for an edge $e = \overset{a}{\bullet} \longrightarrow \overset{b}{\bullet}$ connecting a vertex labeled by $a \in [n]$ and to a vertex labelled by $b \in [n]$, we set

$$\Delta_e(\gamma_1 \otimes \gamma_2 \otimes \dots \otimes \gamma_n) = \begin{cases} \sum_{i \in I} (-1)^{|\psi^i|(|\gamma_a| + |\gamma_{a+1}| + \dots + |\gamma_{b-1}|)} \gamma_1 \otimes \dots \otimes \frac{\partial \gamma_a}{\partial \psi^i} \otimes \dots \otimes \frac{\partial \gamma_b}{\partial \psi^i} \otimes \dots \otimes \gamma_n & \text{for } a < b \\ \sum_{i \in I} (-1)^{|\psi_i|(|\gamma_b| + |\gamma_{b+1}| + \dots + |\gamma_{a-1}| + 1)} \gamma_1 \otimes \dots \otimes \frac{\partial \gamma_b}{\partial \psi^i} \otimes \dots \otimes \frac{\partial \gamma_a}{\partial \psi^i} \otimes \dots \otimes \gamma_n & \text{for } b < a \end{cases}$$

and where μ is the standard multiplication map in the ring $\text{LieB}(\mathfrak{g}) \subset \mathbb{K}[[\psi_i, \eta^i]]$,

$$\begin{array}{ccc} \mu : \text{LieB}(\mathfrak{g})^{\otimes n} & \longrightarrow & \text{LieB}(\mathfrak{g}) \\ \gamma_1 \otimes \gamma_2 \otimes \dots \otimes \gamma_n & \longrightarrow & \gamma_1 \gamma_2 \dots \gamma_n. \end{array}$$

Note that this representation makes sense for both finite- and *infinite* dimensional vector spaces \mathfrak{g} as graphs from $\mathcal{G}ra^\uparrow$ do not contain oriented cycles.

4.3.1. Remark. The above action of $\mathcal{G}ra^\uparrow$ on $\text{Def}(\mathcal{L}ie\mathcal{B}_\infty \xrightarrow{0} \text{End}_{\mathfrak{g}})[-2]$ only uses the properadic compositions in $\text{End}_{\mathfrak{g}}$ and no further data. It follows that the same formulas may in fact be used to define an action of $\mathcal{G}ra^\uparrow$ on the deformation complex

$$\text{Def}(\mathcal{L}ie\mathcal{B}_\infty \xrightarrow{0} \mathcal{P})[-2]$$

for any properad \mathcal{P} .

4.4. An oriented graph complex. Let $\mathcal{L}ie\{2\}$ be a (degree shifted) operad of Lie algebras, and let $\mathcal{L}ie_\infty\{2\}$ be its minimal resolution. Thus $\mathcal{L}ie\{2\}$ is a quadratic operad generated by degree -2 skewsymmetric binary operation,

$$\begin{array}{c} \bullet \\ | \\ \begin{array}{cc} \bullet & \bullet \\ / & \backslash \\ 1 & 2 \end{array} \end{array} = - \begin{array}{c} \bullet \\ | \\ \begin{array}{cc} \bullet & \bullet \\ \backslash & / \\ 2 & 1 \end{array} \end{array}$$

modulo the Jacobi relations,

$$(32) \quad \begin{array}{c} \bullet \\ | \\ \begin{array}{ccc} \bullet & \bullet & \bullet \\ / & \backslash & \backslash \\ 1 & 2 & 3 \end{array} \end{array} + \begin{array}{c} \bullet \\ | \\ \begin{array}{ccc} \bullet & \bullet & \bullet \\ \backslash & / & \backslash \\ 3 & 1 & 2 \end{array} \end{array} + \begin{array}{c} \bullet \\ | \\ \begin{array}{ccc} \bullet & \bullet & \bullet \\ \backslash & / & / \\ 2 & 3 & 1 \end{array} \end{array} = 0$$

while $\mathcal{L}ie_\infty\{2\}$ is the free operad generated by an \mathbb{S} -module $E = \{E(n)\}_{n \geq 2}$,

$$E(n) := \text{sgn}_n[3n - 4] = \left\langle \begin{array}{c} \bullet \\ | \\ \begin{array}{ccccccc} \bullet & \bullet & \bullet & \dots & \bullet & \bullet & \bullet \\ / & \backslash & \backslash & & / & \backslash & / \\ 1 & 2 & 3 & \dots & n-1 & n & \end{array} \end{array} = (-1)^\sigma \begin{array}{c} \bullet \\ | \\ \begin{array}{ccccccc} \bullet & \bullet & \bullet & \dots & \bullet & \bullet & \bullet \\ / & \backslash & \backslash & & / & \backslash & / \\ \sigma(1) & \sigma(2) & \dots & & \sigma(n) & & \end{array} \end{array} \right\rangle_{\sigma \in \mathbb{S}_n}$$

and equipped with the following differential,

$$(33) \quad \partial \begin{array}{c} \bullet \\ | \\ \begin{array}{ccccccc} \bullet & \bullet & \bullet & \dots & \bullet & \bullet & \bullet \\ / & \backslash & \backslash & & / & \backslash & / \\ 1 & 2 & 3 & \dots & n-1 & n & \end{array} \end{array} = \sum_{\substack{[n] = I_1 \sqcup I_2 \\ \#I_1 \geq 1, \#I_2 \geq 1}} (-1)^{\sigma(I_1 \sqcup I_2) + |I_1| |I_2|} \begin{array}{c} \bullet \\ | \\ \begin{array}{c} \bullet \\ | \\ \begin{array}{c} \bullet & \dots & \bullet \\ / & & \backslash \\ I_1 & & I_2 \end{array} \end{array} \end{array}$$

where $\sigma(I_1 \sqcup I_2)$ is the sign of the shuffle $[n] \rightarrow [I_1 \sqcup I_2]$.

4.4.1. Proposition [W3]. *There is a morphism of operads*

$$\varphi : \mathcal{L}ie\{2\} \longrightarrow \mathcal{G}ra^\uparrow$$

given on the generators by

$$\begin{array}{c} | \\ \bullet \\ / \quad \backslash \\ 1 \quad 2 \end{array} \longrightarrow \bullet \xrightarrow{1} \bullet \xrightarrow{2} \bullet - \bullet \xrightarrow{2} \bullet \xrightarrow{1} \bullet$$

Proof. Using the definition of the operadic composition in $\mathcal{G}ra^\uparrow$ we get

$$(34) \quad \varphi \left(\begin{array}{c} | \\ \bullet \\ / \quad \backslash \\ 1 \quad 2 \\ \bullet \\ / \quad \backslash \\ 3 \end{array} \right) = \begin{array}{c} \bullet \xrightarrow{1} \bullet \xrightarrow{2} \bullet \xrightarrow{3} \bullet \\ \bullet \xrightarrow{2} \bullet \xrightarrow{1} \bullet \xrightarrow{3} \bullet \\ \bullet \xrightarrow{2} \bullet \xrightarrow{1} \bullet \xrightarrow{3} \bullet \\ \bullet \xrightarrow{1} \bullet \xrightarrow{2} \bullet \xrightarrow{3} \bullet \\ \bullet \xleftarrow{1} \bullet \xleftarrow{2} \bullet \xleftarrow{3} \bullet \\ \bullet \xleftarrow{2} \bullet \xleftarrow{1} \bullet \xleftarrow{3} \bullet \\ \bullet \xrightarrow{2} \bullet \xleftarrow{1} \bullet \xrightarrow{3} \bullet \\ \bullet \xrightarrow{1} \bullet \xrightarrow{2} \bullet \xleftarrow{3} \bullet \end{array}$$

which implies

$$\varphi \left(\begin{array}{c} | \\ \bullet \\ / \quad \backslash \\ 1 \quad 2 \\ \bullet \\ / \quad \backslash \\ 3 \end{array} + \begin{array}{c} | \\ \bullet \\ / \quad \backslash \\ 3 \quad 1 \\ \bullet \\ / \quad \backslash \\ 2 \end{array} + \begin{array}{c} | \\ \bullet \\ / \quad \backslash \\ 2 \quad 3 \\ \bullet \\ / \quad \backslash \\ 1 \end{array} \right) = 0.$$

□

All possible morphisms of dg operads, $\mathcal{L}ie_\infty\{2\} \longrightarrow \mathcal{G}ra^\uparrow$, can be usefully encoded as Maurer-Cartan elements in the graded Lie algebra,

$$\mathfrak{fGC}_3^{or} := \text{Def}(\mathcal{L}ie_\infty\{2\} \xrightarrow{0} \mathcal{G}ra^\uparrow),$$

which controls deformation theory of the zero morphism (cf. [MeVa]). As a graded vector space,

$$(35) \quad \mathfrak{fGC}_3^{or} \cong \prod_{n \geq 2} \text{Hom}_{\mathbb{S}_n}(E(n), \mathcal{G}ra^\uparrow(n))[-1] = \prod_{n \geq 2} \mathcal{G}ra^\uparrow(n)^{\mathbb{S}_n}[3 - 3n],$$

so that its elements can be understood as (\mathbb{K} -linear series of) graphs Γ from $\mathcal{G}ra^\uparrow$ whose vertex labels are skewsymmetrized (so that we can often forget numerical labels of vertices in our pictures), and which are assigned the homological degree

$$|\Gamma| = 3\#V(\Gamma) - 3 - 2\#E(\Gamma),$$

where $V(\Gamma)$ (resp. $E(\Gamma)$) stands for the set of vertices (resp., edges) of Γ .

The Lie brackets, $[\cdot, \cdot]_{\text{gra}}$, in \mathfrak{fGC}_3^{or} can be either read from the differential (33), or, equivalently, from the following explicit Lie algebra structure [KM] associated with the degree shifted operad $\mathcal{G}ra_3^\uparrow\{3\}$ (and which makes sense for any operad),

$$[\cdot, \cdot] : \begin{array}{ccc} \mathbb{P} \otimes \mathbb{P} & \longrightarrow & \mathbb{P} \\ (a \in \mathcal{P}(n), b \in \mathcal{P}(m)) & \longrightarrow & [a, b] := \sum_{i=1}^n a \circ_i b - (-1)^{|a||b|} \sum_{i=1}^m b \circ_i a \end{array}$$

where $\mathbb{P} := \prod_{n \geq 1} \mathcal{G}ra(n)^\uparrow[3 - 3n]$. These Lie brackets in \mathbb{P} induce Lie brackets, $[\cdot, \cdot]_{\text{gra}}$, in the subspace of \mathbb{S} -invariants [KM],

$$\mathbb{P}^{\mathbb{S}} := \prod_{n \geq 1} \mathcal{G}ra^\uparrow(n)[3 - 3n]^{\mathbb{S}_n} = \mathfrak{fGC}_3^{or}$$

via the standard symmetrization map $\mathbb{P} \rightarrow \mathbb{P}^{\mathbb{S}}$.

The graph

$$\bullet \xrightarrow{1} \bullet \xrightarrow{2} \bullet - \bullet \xrightarrow{2} \bullet \xrightarrow{1} \bullet$$

is a degree $2 \cdot 3 - 3 - 2 = 1$ element in \mathfrak{GC}_3^{or} , which, in fact, is a Maurer-Cartan element,

$$[\bullet \xrightarrow{1} \bullet, \bullet \xrightarrow{2} \bullet] = \text{skewsymmetrization of the r.h.s. in (34)} = 0,$$

which represents the above morphism φ in the Lie algebra \mathfrak{fGC}_3^{or} . This element makes, therefore, \mathfrak{fGC}_3^{or} into a *differential* graded Lie algebra with the differential


$$(36) \quad d\Gamma := [\bullet \xrightarrow{1} \bullet, \Gamma]_{\text{gra}}.$$

Let \mathfrak{GC}_3^{or} be a subspace of \mathfrak{fGC}_3^{or} spanned by connected graphs whose vertices are at least bivalent, and if bivalent do not have one incoming and one outgoing edge. It is easy to see that this is a dg Lie subalgebra.

4.4.1. Remark. This definition of GC_3^{or} in [W1] differs slightly from the present one in that all bivalent vertices are allowed in loc. cit. However, it is easy to check that this extra condition does not change the cohomology.

The cohomology of the *oriented graph complex* (GC_3^{or}, d) was partially computed in [W1].

4.4.2. Theorem [W3]. (i) $H^0(\mathrm{GC}_3^{or}, d) = \mathfrak{grt}_1$, where \mathfrak{grt}_1 is the Lie algebra of the pronipotent Grothendieck-Teichmüller group GRT_1 introduced by Drinfeld in [D2].

(ii) $H^{-1}(\mathrm{GC}_3^{or}, d) \cong \mathbb{K}$. The single class is represented by the graph .

(iii) $H^i(\mathrm{GC}_3^{or}, d) = 0$ for all $i \leq -2$.

4.5. Action on $\widehat{\mathcal{L}ie\mathcal{B}}_\infty$. There is a natural action of GC_3^{or} on the properad $\widehat{\mathcal{L}ie\mathcal{B}}_\infty$ by properadic derivations. Concretely, for any graph Γ we define the derivation $F(\Gamma) \in \mathrm{Der}(\widehat{\mathcal{L}ie\mathcal{B}}_\infty)$ sending the generator $\mu_{m,n}$ of $\widehat{\mathcal{L}ie\mathcal{B}}_\infty$ to the linear combination of graphs

$$(37) \quad \mu_{m,n} \cdot \Gamma = \sum \underbrace{\begin{array}{c} m \times \\ \downarrow \cdots \downarrow \\ \Gamma \\ \uparrow \cdots \uparrow \\ n \times \end{array}}_{n \times}$$

where the sum is taken over all ways of attaching the incoming and outgoing legs such that all vertices are at least trivalent and have at least one incoming and one outgoing edge.

4.5.1. Lemma. *The above formula defines a right action of GC_3^{or} on $\widehat{\mathcal{L}ie\mathcal{B}}_\infty$.*

Proof sketch. We denote by \bullet the pre-Lie product on the deformation complex $\mathrm{Def}(\mathcal{L}ie_\infty\{2\} \xrightarrow{0} \mathrm{Gra}^\uparrow) \supset \mathrm{GC}_3^{or}$, so that the Lie bracket on GC_3^{or} may be written as $[\Gamma, \Gamma'] = \Gamma \bullet \Gamma' \pm \Gamma' \bullet \Gamma$ for $\Gamma, \Gamma' \in \mathrm{GC}_3^{or}$. Note that

$$\mu_{m,n} \cdot (\Gamma \bullet \Gamma') = (\mu_{m,n} \cdot \Gamma) \cdot \Gamma',$$

where it is important that we excluded graphs with bivalent vertices with one incoming and one outgoing edge from the definition of GC_3^{or} . It follows that the formula above defines an action of the graded Lie algebra GC_3^{or} . We leave it to the reader to check that this action also commutes with the differential. \square

Of course, by a change of sign the right action may be transformed into a left action and hence we obtain a map of Lie algebras

$$F: \mathrm{GC}_3^{or} \rightarrow \mathrm{Der}(\widehat{\mathcal{L}ie\mathcal{B}}_\infty).$$

Interpreting the right hand side as a graph complex as in section 3.2, the map F sends a graph $\Gamma \in \mathrm{GC}_3^{or}$ to the series of graphs

$$\pm \sum_{m,n} \sum \underbrace{\begin{array}{c} m \times \\ \downarrow \cdots \downarrow \\ \Gamma \\ \uparrow \cdots \uparrow \\ n \times \end{array}}_{n \times}$$

4.5.2. Theorem. *The map $F: \mathrm{GC}_3^{or} \rightarrow \mathrm{Der}(\widehat{\mathcal{L}ie\mathcal{B}}_\infty)$ is a quasi-isomorphism, up to one class in $\mathrm{Der}(\widehat{\mathcal{L}ie\mathcal{B}}_\infty)$ represented by the series*

$$\sum_{m,n} (m+n-2) \underbrace{\begin{array}{c} m \times \\ \downarrow \cdots \downarrow \\ \bullet \\ \uparrow \cdots \uparrow \\ n \times \end{array}}_{n \times}.$$

The result will not be used directly in this paper. The proof is an adaptation of the proof of [W3, Proposition 3] and is given in Appendix B.1.

4.6. GRT_1 action on Lie bialgebra structures. The action of the Lie algebra of closed degree zero cocycles $GC_{3,cl}^{or} \subset GC_3^{or}$ on \widehat{LieB}_∞ by derivations may be integrated to an action of the exponential group $\text{Exp } GC_{3,cl}^{or}$ on \widehat{LieB}_∞ by (continuous) automorphisms. Hence this exponential group acts on the set of \widehat{LieB}_∞ algebra structures on any dg vector space \mathfrak{g} , i. e., on the set of morphisms of properads

$$\widehat{LieB}_\infty \rightarrow \text{End}_{\mathfrak{g}}$$

by precomposition. Furthermore, it follows that the cohomology Lie algebra $H^0(GC_3^{or}) \cong \mathfrak{grt}_1$ maps into the the Lie algebra of continuous derivations up to homotopy $H^0(\text{Der}(\widehat{LieB}_\infty))$ and the the exponential group $\text{Exp } H^0(GC_3^{or}) \cong GRT_1$ maps into the set of homotopy classes of continuous automorphisms of \widehat{LieB}_∞ .

4.6.1. Remark. Note also that one may define a non-complete version $GC_{3,inc}^{or}$ of the graph complex GC_3^{or} by merely replacing the direct product by a direct sum in (35). The zeroth cohomology of $GC_{3,inc}^{or}$ is a non-complete version of the Grothendieck-Teichmüller group. Furthermore $GC_{3,inc}^{or}$ acts on the non-completed operad $LieB_\infty$ by derivations, using the formulas of the previous subsection, and hence also on $LieB_\infty$ algebra structures. However, these actions can in general not be integrated, whence we work with the completed properad \widehat{LieB}_∞ above.

Finally, let us describe the action $GC_{3,cl}^{or}$ on Lie bialgebra structures in yet another form. We have a sequence of morphisms of dg Lie algebras,

$$GC_3^{or} \longrightarrow \text{Def}(Lie_\infty\{2\} \rightarrow \text{Gra}^\uparrow) \xrightarrow{\rho} \text{Def}(Lie_\infty\{2\} \xrightarrow{\{ \cdot, \cdot \}} \mathcal{E}nd_{\text{LieB}(\mathfrak{g})})$$

where the first arrow is just the inclusion, and the second arrow is induced by the canonical representation (31) and which obviously satisfies

$$\rho \circ \varphi \left(\begin{array}{c} \bullet \\ / \quad \backslash \\ 1 \quad 2 \end{array} \right) = \{ \cdot, \cdot \} \in \text{Hom}(\wedge^2 \text{LieB}, \text{LieB}[-2]) \subset \text{Def}(Lie_\infty\{2\} \xrightarrow{\{ \cdot, \cdot \}} \mathcal{E}nd_{\text{LieB}(\mathfrak{g})}).$$

The dg Lie algebra,

$$\text{Def}(Lie_\infty\{2\} \xrightarrow{\{ \cdot, \cdot \}} \mathcal{E}nd_{\text{LieB}(\mathfrak{g})}) =: CE^\bullet(\text{LieB}(\mathfrak{g})),$$

is nothing but the classical Chevalley-Eilenberg complex controlling deformations of the Poisson brackets (28) in $\text{LieB}(\mathfrak{g})$. In particular for any closed degree zero element $g \in GC_3^{or}$, and in particular for representatives of elements of \mathfrak{grt}_1 in GC_3^{or} , we obtain a Lie_∞ derivation of $\text{LieB}(\mathfrak{g})$. This derivation may be integrated into a Lie_∞ automorphism $\exp(ad_g)$. For a Maurer-Cartan element γ in $\text{LieB}(\mathfrak{g})$ corresponding to a graded complete $LieB_\infty$ structure on \mathfrak{g} the series

$$\gamma \longrightarrow \exp(ad_g)\gamma$$

converges and defines again a graded complete homotopy Lie bialgebra structure on \mathfrak{g} .

4.6.2. Remark. By degree reasons the above action on \widehat{LieB}_∞ structures maps \widehat{LieB} structures again to \widehat{LieB} structures. In other words, no higher homotopies are created if there were none before.

4.7. Another oriented graph complex. We shall introduce next a new oriented graph complex and then use (both results of) Theorem 4.4.2 to compute partially its cohomology and then deduce formulae for an action of GRT_1 on *involutive* Lie bialgebra structures.

Let \hbar be a formal variable of homological degree 2. The Lie brackets $[\cdot, \cdot]_{\text{gra}}$ in GC_3^{or} extend \hbar -linearly to the topological vector space $GC_3^{or}[[\hbar]]$.

4.8. Proposition. *The element*

$$\Phi_{\hbar} := \sum_{k=1}^{\infty} \hbar^{k-1} \underbrace{\begin{array}{c} \text{---} \\ \circ \\ \text{---} \\ \text{---} \\ \circ \\ \text{---} \end{array}}_{k \text{ edges}}$$

is a Maurer-Cartan element in the Lie algebra $(\mathfrak{f}GC_3^{or}[[\hbar]], [\cdot, \cdot]_{\text{gra}})$.

Proof.

$$\begin{aligned}
[\Phi_{\hbar}, \Phi_{\hbar}] &= \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \hbar^{k+l-2} \sum_{k=k'+k''} \left(\text{Diagram 1} \right) - \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \hbar^{k+l-2} \sum_{k=k'+k''} \left(\text{Diagram 2} \right) \\
&= 0.
\end{aligned}$$

□

Hence the degree one continuous map

$$\begin{aligned}
d_{\hbar} : \text{GC}_3^{or}[[\hbar]] &\longrightarrow \text{GC}_3^{or}[[\hbar]] \\
\Gamma &\longrightarrow d_{\hbar}\Gamma := [\Phi_{\hbar}, \Gamma]_{\text{gra}}
\end{aligned}$$

is a differential in $\text{GC}_3^{or}[[\hbar]]$. The induced differential, d , in $\text{GC}_3^{or} = \text{GC}_3^{or}[[\hbar]]/\hbar\text{GC}_3^{or}[[\hbar]]$ is precisely the original differential (36).

4.9. Action on $\widehat{\mathcal{L}ie^{\circ}\mathcal{B}_{\infty}}$. The dg Lie algebra $(\text{GC}_3^{or}[[\hbar]], d_{\hbar})$ acts naturally on the properad $\widehat{\mathcal{L}ie^{\circ}\mathcal{B}_{\infty}}$ by continuous properadic derivations. More precisely, let $\Gamma \in \text{GC}_3^{or}$ be a graph. Then to the element $\hbar^N \Gamma \in \text{GC}_3^{or}[[\hbar]]$ we assign the derivation of $\widehat{\mathcal{L}ie^{\circ}\mathcal{B}_{\infty}}$ that sends the generator $\mu_{m,n}^k$ to zero if $k < N$ and to

$$(38) \quad \mu_{m,n}^k \cdot (\hbar^N \Gamma) = \sum \left(\text{Diagram 3} \right)$$

where the sum is over all graphs obtained by (i) attaching the external legs to Γ in all possible ways as in (38) and (ii) assigning weights to the vertices in all possible ways such that the weights sum to $k - N$.

4.9.1. Lemma. *The above formula defines a right action of $(\text{GC}_3^{or}[[\hbar]], d_{\hbar})$ on $\widehat{\mathcal{L}ie^{\circ}\mathcal{B}_{\infty}}$.*

Proof sketch. The proof is similar to that of Lemma 4.5.1 after noting that

$$\mu_{m,n}^k \cdot (\hbar^N \Gamma \bullet \hbar^M \Gamma') = (\mu_{m,n} \cdot \hbar^N \Gamma) \cdot \hbar^M \Gamma'$$

for all M, N and $\Gamma, \Gamma' \in \text{GC}_3^{or}$. □

Again, by a change of sign the right action may be transformed into a left action and hence we obtain a map of Lie algebras

$$F_{\hbar} : \text{GC}_3^{or}[[\hbar]] \longrightarrow \text{Der}(\widehat{\mathcal{L}ie^{\circ}\mathcal{B}_{\infty}}).$$

4.9.2. Theorem. *The map F_{\hbar} is a quasi-isomorphism, up to classes $T\mathbb{K}[[\hbar]] \subset \text{Der}(\widehat{\mathcal{L}ie^{\circ}\mathcal{B}_{\infty}})$ where*

$$T = \sum_{m,n,p} (m+n+2p-2)\hbar^p \left(\text{Diagram 4} \right).$$

The result will not be used in this paper, and the proof will be given in Appendix B.2.

4.10. Action on involutive Lie bialgebra structures. By the previous subsection the Lie algebra of degree 0 cocycles in $\text{GC}_3^{or}[[\hbar]]$ acts on the properad $\widehat{\mathcal{L}ie^{\circ}\mathcal{B}_{\infty}}$ by derivations. The action may be integrated to an action of the corresponding exponential group on $\widehat{\mathcal{L}ie^{\circ}\mathcal{B}_{\infty}}$ by continuous automorphisms, and hence also on the set of $\widehat{\mathcal{L}ie^{\circ}\mathcal{B}_{\infty}}$ algebra structures on some dg vector space by precomposition. Furthermore, the cohomology Lie algebra $H^0(\text{GC}_3^{or}[[\hbar]])$ maps into the the Lie algebra of continuous derivations up to homotopy $H^0(\text{Der}(\widehat{\mathcal{L}ie^{\circ}\mathcal{B}_{\infty}}))$, while the the exponential group $\exp H^0(\text{GC}_3^{or}[[\hbar]])$ maps into the set of homotopy classes of continuous automorphisms of $\widehat{\mathcal{L}ie^{\circ}\mathcal{B}_{\infty}}$. By precomposition, we also have a map of Lie algebras $H^0(\text{GC}_3^{or}[[\hbar]]) \rightarrow H^0(\text{Def}(\widehat{\mathcal{L}ie^{\circ}\mathcal{B}_{\infty}} \rightarrow \text{End}_{\mathfrak{g}}))$ for any $\widehat{\mathcal{L}ie^{\circ}\mathcal{B}_{\infty}}$ algebra \mathfrak{g} and an action of $\exp H^0(\text{GC}_3^{or}[[\hbar]])$ on the set of such algebra structures up to homotopy. Let us encode these findings in the following corollary.

4.10.1. Corollary. *The Lie algebra $H^0(\mathrm{GC}_3^{or}[[\hbar]])$ and its exponential group $\exp(H^0(\mathrm{GC}_3^{or}[[\hbar]]))$ canonically act on the set of graded complete strong homotopy involutive Lie bialgebra structures up to homotopy.*

4.10.2. Remark. Note again that, analogously to Remark 4.6.1, we may define a non-complete version of the graph complex $\mathrm{GC}_3^{or}[[\hbar]]$ which acts on the non-complete properad $\mathcal{L}ie^\circ\mathcal{B}_\infty$ by derivations, and hence also on ordinary Lie bialgebra structures. However, these actions can in general not be integrated, whence we prefer to work with the complete version of the graph complex and $\widehat{\mathcal{L}ie^\circ\mathcal{B}_\infty}$.

Finally let us give another yet another description of the action of $\mathrm{GC}_3^{or}[[\hbar]]$ on involutive Lie bialgebra structures, strengthening the above result a little. As usual, the deformation complex of the zero morphism of dg props, $\mathrm{InvLieB}(\mathfrak{g}) := \mathrm{Def}(\mathcal{L}ie^\circ\mathcal{B}_\infty \xrightarrow{0} \mathcal{E}nd_{\mathfrak{g}})$ has a canonical dg Lie algebra structure, with the Lie bracket $[\cdot, \cdot]_{\hbar}$ given explicitly by (25), such that the Maurer-Cartan elements are in 1-to-1 correspondence with $\mathcal{L}ie^\circ\mathcal{B}_\infty$ -structures on the dg vector space \mathfrak{g} . The Maurer-Cartan element Φ_{\hbar} in $\mathrm{fGC}_3^{or}[[\hbar]]$ corresponds to a continuous morphism of operads,

$$\varphi_{\hbar} : \mathcal{L}ie\{2\}[[\hbar]] \longrightarrow \mathcal{G}ra^{\uparrow}[[\hbar]],$$

given on the generator of $\mathcal{L}ie\{2\}$ by the formula

$$\varphi_{\hbar} \left(\begin{array}{c} \bullet \\ | \\ \begin{array}{cc} 1 & 2 \end{array} \end{array} \right) = \sum_{k=1}^{\infty} \hbar^{k-1} \left(\begin{array}{c} 1 \\ \circlearrowleft \scriptstyle (k) \\ \bullet \\ \circlearrowright \scriptstyle (k) \\ 2 \end{array} \right) - \begin{array}{c} 2 \\ \circlearrowleft \scriptstyle (k) \\ \bullet \\ \circlearrowright \scriptstyle (k) \\ 1 \end{array} \right).$$

The representation (31) of the operad $\mathcal{G}ra^{\uparrow}$ in the deformation complex $\mathrm{Def}(\mathcal{L}ie\mathcal{B}_\infty \xrightarrow{0} \mathcal{P})[-2]$ extends \hbar -linearly to a representation ρ_{\hbar} of $\mathcal{G}ra^{\uparrow}[[\hbar]]$ in $\mathrm{Def}(\mathcal{L}ie^\circ\mathcal{B}_\infty \xrightarrow{0} \mathcal{P})[-2]$ for any properad \mathcal{P} . Furthermore it is almost immediate to see that the action of $\mathcal{L}ie\{2\}$ on the latter deformation complex factors through the map $\mathcal{L}ie\{2\} \rightarrow \mathcal{G}ra^{\uparrow}[[\hbar]]$.

It follows that one has a morphism of dg Lie algebras induced by φ_{\hbar}

$$(\mathrm{GC}_3^{or}[[\hbar]], d_{\hbar}) \longrightarrow \mathrm{Def}(\mathcal{L}ie_{\infty}\{2\}[[\hbar]] \xrightarrow{\varphi_{\hbar}} \mathcal{G}ra^{\uparrow}[[\hbar]]) \xrightarrow{\rho_{\hbar}} \mathrm{Def}(\mathcal{L}ie_{\infty}\{2\}[[\hbar]] \xrightarrow{[\cdot, \cdot]_{\hbar}} \mathcal{E}nd_D) =: CE^{\bullet}(D)$$

from the graph complex $(\mathrm{GC}_3^{or}[[\hbar]], d_{\hbar})$ into the Chevalley-Eilenberg dg Lie algebra of $D := \mathrm{Def}(\mathcal{L}ie\mathcal{B}_\infty \xrightarrow{0} \mathcal{P})[-2]$. In particular this implies the following:

4.11. Theorem. *$H^0(\mathrm{GC}_3^{or}[[\hbar]], d_{\hbar}) \simeq H^0(\mathrm{GC}_3^{or}, d_0) \simeq \mathfrak{grt}_1$ as Lie algebras. Moreover, $H^i(\mathrm{GC}_3^{or}[[\hbar]], d_{\hbar}) = 0$ for all $i \leq -2$ and $H^{-1}(\mathrm{GC}_3^{or}[[\hbar]], d_{\hbar}) \cong \mathbb{K}$, with the single class being represented by*

$$\sum_{k=2}^{\infty} (k-1) \hbar^{k-2} \underbrace{\begin{array}{c} \bullet \\ \circlearrowleft \scriptstyle (k) \\ \bullet \\ \circlearrowright \scriptstyle (k) \\ \bullet \end{array}}_{k \text{ edges}}$$

Proof. First note that the element above is exactly $\frac{d}{d\hbar}\Phi_{\hbar}$ and the fact that it is closed follows easily by differentiating the Maurer-Cartan equation

$$0 = \frac{d}{d\hbar}[\Phi_{\hbar}, \Phi_{\hbar}]_{\mathrm{gra}} = 2[\Phi_{\hbar}, \frac{d}{d\hbar}\Phi_{\hbar}]_{\mathrm{gra}} = 2d_{\hbar} \left(\frac{d}{d\hbar}\Phi_{\hbar} \right).$$

It is easy to see that the cocycle $\frac{d}{d\hbar}\Phi_{\hbar}$ cannot be exact, by just considering the leading term in \hbar . Let us write

$$d_{\hbar} = \sum_{k=1}^{\infty} \hbar^{k-1} d_k.$$

Consider a decreasing filtration of $\mathrm{GC}_3^{or}[[\hbar]]$ by the powers in \hbar . The first term of the associated spectral sequence is

$$\mathcal{E}_1 = \bigoplus_{i \in \mathbb{Z}} \mathcal{E}_1^i, \quad \mathcal{E}_1^i = \bigoplus_{p \geq 0} H^{i-2p}(\mathrm{GC}_3^{or}, d_0) \hbar^p$$

with the differential equal to $\hbar d_1$. The main result of [W3] states that $H^0(\mathrm{GC}_3^{or}, d_0) \simeq \mathfrak{grt}_1$, $H^{\leq -2}(\mathrm{GC}_3^{or}, d_0) = 0$ and $H^{-1}(\mathrm{GC}_3^{or}, d_0) \cong \mathbb{K}$. The desired results follow by degree reasons. \square

4.11.1. Corollary. *The group GRT_1 acts on the set of all possible $\text{Lie}^\circ\mathcal{B}_\infty$ -structures up to homotopy on an arbitrary (perhaps infinite-dimensional) differential graded vector space \mathfrak{g} .*

4.11.2. Iterative construction of graph representatives of elements of $\mathfrak{g}t_1$. The above Theorem 4.11 says that any degree zero graph $\Gamma \in \text{GC}_3^{or}$ satisfying the cocycle condition, $d_0\Gamma_0 = 0$, can be extended to a formal power series,

$$\Gamma_{\hbar} = \Gamma_0 + \hbar\Gamma_1 + \hbar^2\Gamma_2 + \dots,$$

satisfying the cocycle condition $d_{\hbar}\Gamma_{\hbar} = 0$. Let us show how this inductive extension works in detail. The equation $d_{\hbar}^2 = 0$ implies, for any $n \geq 0$, $\sum_{i,j \geq 0}^{n=i+j} d_i d_j = 0$, which in turn reads,

$$\begin{aligned} d_0^2 &= 0 \\ d_0 d_1 + d_1 d_0 &= 0 \\ d_0 d_2 + d_2 d_0 + d_1^2 &= 0 \quad \text{etc.} \end{aligned}$$

Thus the equation $d_0\Gamma_0 = 0$ implies

$$0 = d_1 d_0 \Gamma_0 = -d_0 d_1 \Gamma_0$$

The oriented graph $d_1\Gamma_0 \in \text{GC}_3^{or}$ has degree -1 and $H^{-1}(\text{GC}_3^{or}, d_0) \cong \mathbb{K}$. Since the one cohomology class cannot be hit (its leading term has necessarily only two vertices), there exists a degree -2 graph Γ_1 such that $d_1\Gamma_0 = -d_0\Gamma_1$ so that

$$d_{\hbar}(\Gamma_0 + \hbar\Gamma_1) = 0 \text{ mod } \hbar^2$$

Assume by induction that we constructed a degree zero polynomial,

$$\Gamma_0 + \hbar\Gamma_1 + \dots + \hbar^n\Gamma_n \in \text{GC}_3^{or}[[\hbar]]$$

such that

$$(39) \quad d_{\hbar}(\Gamma_0 + \hbar\Gamma_1 + \dots + \hbar^n\Gamma_n) = 0 \text{ mod } \hbar^{n+1}.$$

Let us show that there exists an oriented graph Γ_{n+1} of degree $-2n-2$ such that

$$d_{\hbar}(\Gamma_0 + \hbar\Gamma_1 + \dots + \hbar^n\Gamma_n + \hbar^{n+1}\Gamma_{n+1}) = 0 \text{ mod } \hbar^{n+2}.$$

or, equivalently, such that

$$(40) \quad d_0\Gamma_{n+1} + d_{n+1}\Gamma_0 + \sum_{\substack{n+1=i+j \\ i,j \geq 1}} d_i\Gamma_j = 0.$$

Equation (39) implies, for any $j \leq n$,

$$d_0\Gamma_j + d_j\Gamma_0 + \sum_{\substack{j=p+q \\ p,q \geq 1}} d_p\Gamma_q = 0.$$

We have

$$\begin{aligned} 0 &= d_{n+1}d_0\Gamma_0 = -d_0d_{n+1}\Gamma_0 - \sum_{\substack{n+1=i+j \\ i,j \geq 1}} d_i d_j \Gamma_0 \\ &= -d_0d_{n+1}\Gamma_0 + \sum_{\substack{n+1=i+j \\ i,j \geq 1}} d_i d_0 \Gamma_j + \sum_{\substack{n+1=i+p+q \\ i,p,q \geq 1}} d_i d_p \Gamma_q \\ &= -d_0d_{n+1}\Gamma_0 - \sum_{\substack{n+1=i+j \\ i,j \geq 1}} d_0 d_i \Gamma_j - \sum_{\substack{n+1=i+p+q \\ i,p,q \geq 1}} d_i d_p \Gamma_q + \sum_{\substack{n+1=i+p+q \\ i,p,q \geq 1}} d_i d_p \Gamma_q \\ &= -d_0 \left(d_{n+1}\Gamma_0 + \sum_{\substack{n+1=i+j \\ i,j \geq 1}} d_i \Gamma_j \right) \end{aligned}$$

As $H^{-1-2n}(\text{GC}_3^{or}, d_0) = 0$ for all $n \geq 1$, there exists a degree $-2-2n$ graph Γ_{n+1} such that the required equation (40) is satisfied. This completes an inductive construction of Γ_{\hbar} from Γ_0 .

4.12. Deformations of Frobenius algebra structures. Note that the complexes $\text{Def}(\mathcal{F}rob_\infty^\diamond \rightarrow \mathcal{F}rob^\diamond)$ and $\text{Def}(\mathcal{L}ie\mathcal{B}_\infty \rightarrow \widehat{\mathcal{L}ie\mathcal{B}})$ are isomorphic. We hence have a zigzag of (quasi-)isomorphisms of complexes

$$\text{Der}(\mathcal{F}rob_\infty^\diamond) \rightarrow \text{Def}(\mathcal{F}rob_\infty^\diamond \rightarrow \mathcal{F}rob^\diamond)[1] \cong \text{Def}(\mathcal{L}ie\mathcal{B}_\infty \rightarrow \widehat{\mathcal{L}ie\mathcal{B}})[1] \leftarrow \text{Der}(\widehat{\mathcal{L}ie\mathcal{B}}_\infty).$$

In particular we obtain a map⁴

$$\mathbf{grt}_1 \rightarrow H^0(\text{Der}(\widehat{\mathcal{L}ie\mathcal{B}}_\infty)) \cong H^0(\text{Der}(\mathcal{F}rob_\infty^\diamond)).$$

Hence we obtain a large class of homotopy non-trivial derivations of the properad $\mathcal{F}rob_\infty^\diamond$ and accordingly a large class of potentially homotopy non-trivial universal deformations of any $\mathcal{F}rob_\infty^\diamond$ algebra.

4.12.1. Corollary. *There is a map $\mathbf{grt}_1 \rightarrow H^0(\text{Der}(\mathcal{F}rob_\infty^\diamond))$ and hence a map $\mathbf{grt}_1 \rightarrow H^1(\text{Def}(\mathcal{F}rob_\infty^\diamond \rightarrow \text{End}_A))$ for any $\mathcal{F}rob_\infty^\diamond$ algebra A .*

Next consider the Frobenius properad $\mathcal{F}rob$ and let $\widehat{\mathcal{F}rob}$ be its genus completion. Analogously to section 4.2 let $\widehat{\mathcal{F}rob}_\infty$ be the completion of $\mathcal{F}rob_\infty$ with respect to the total genus and let $\text{Der}(\widehat{\mathcal{F}rob}_\infty)$ be the continuous derivations. Note that the complex $\text{Def}(\mathcal{F}rob_\infty \rightarrow \widehat{\mathcal{F}rob})$ is isomorphic to the complex $\text{Def}(\mathcal{L}ie^\diamond\mathcal{B}_\infty \rightarrow \widehat{\mathcal{L}ie^\diamond\mathcal{B}})$. We hence obtain a zigzag of quasi-isomorphisms

$$\text{Der}(\widehat{\mathcal{F}rob}_\infty) \rightarrow \text{Def}(\mathcal{F}rob_\infty \rightarrow \widehat{\mathcal{F}rob})[1] \cong \text{Def}(\mathcal{L}ie^\diamond\mathcal{B}_\infty \rightarrow \widehat{\mathcal{L}ie^\diamond\mathcal{B}})[1] \leftarrow \text{Der}(\widehat{\mathcal{L}ie^\diamond\mathcal{B}}_\infty).$$

In particular we obtain a map

$$\mathbf{grt}_1 \rightarrow H^0(\text{Der}(\widehat{\mathcal{L}ie^\diamond\mathcal{B}}_\infty)) \cong H^0(\text{Der}(\widehat{\mathcal{F}rob}_\infty)).$$

Consider the explicit construction of representatives of \mathbf{grt}_1 -elements of section 4.11.2. The \hbar^n -correction term Γ_n to some graph cohomology class Γ of genus g has genus $g + n$. It follows that the map $\mathbf{grt}_1 \rightarrow H^0(\text{Der}(\widehat{\mathcal{F}rob}_\infty))$ in fact factors through $H^0(\text{Der}(\mathcal{F}rob_\infty))$ and in particular we have a map

$$\mathbf{grt}_1 \rightarrow H^0(\text{Der}(\mathcal{F}rob_\infty))$$

and hence a map from \mathbf{grt}_1 into the deformation complex of any $\mathcal{F}rob_\infty$ algebra.

4.12.2. Corollary. *There is a map $\mathbf{grt}_1 \rightarrow H^0(\text{Der}(\mathcal{F}rob_\infty))$ and hence a map $\mathbf{grt}_1 \rightarrow H^1(\text{Def}(\mathcal{F}rob_\infty \rightarrow \text{End}_A))$ for any $\mathcal{F}rob_\infty$ algebra A .*

5. Involutive Lie bialgebras as homotopy Batalin-Vilkovisky algebras

Let \mathfrak{g} be a $\mathcal{L}ie\mathcal{B}$ algebra. Then it is a well known fact the Chevalley-Eilenberg complex of \mathfrak{g} (as a Lie coalgebra) $CE(\mathfrak{g}) = \odot^\bullet\mathfrak{g}[-1]$ carries a Gerstenhaber algebra structure. Concretely, the commutative algebra structure is the obvious one. To define the Lie bracket it is sufficient to define it on the generators $\mathfrak{g}[-1]$, where it is given by the Lie bracket on \mathfrak{g} .

Similarly, if \mathfrak{g} is an $\mathcal{L}ie^\diamond\mathcal{B}$ algebra, then $CE(\mathfrak{g}) = \odot^\bullet\mathfrak{g}[-1]$ carries a natural Batalin-Vilkovisky (BV) algebra structure. The product and Lie bracket are as before. The BV operator Δ is defined on a word $x_1 \cdots x_n$ as

$$\Delta(x_1 \cdots x_n) = - \sum_{i < j} (-1)^{i+j} [x_i, x_j] x_1 \cdots \hat{x}_i \cdots \hat{x}_j \cdots x_n.$$

The involutivity condition is needed for the BV operator to be compatible with the differential. Now suppose that \mathfrak{g} is a $\mathcal{L}ie\mathcal{B}_\infty$ algebra. We call it *good* if for any fixed m only finitely many of the operations $\mu_{m,n} \in \text{Hom}(\mathfrak{g}^{\otimes m}, \mathfrak{g}^{\otimes n})$ are non-zero. Then one may define the Chevalley complex $CE(\mathfrak{g}) = \odot^\bullet\mathfrak{g}[-1]$ of \mathfrak{g} as an $\mathcal{L}ie_\infty$ coalgebra. It is known (see, e. g. Remark 1 of [W3]) that $CE(\mathfrak{g}) = \odot^\bullet\mathfrak{g}[-1]$ carries a natural homotopy Gerstenhaber structure. In this section we show that similarly, if \mathfrak{g} is a good $\mathcal{L}ie^\diamond\mathcal{B}_\infty$ algebra, then the Chevalley-Eilenberg complex $CE(\mathfrak{g})$ carries a natural homotopy BV algebra structure.

⁴In fact, the first arrow is an injection and almost an isomorphism by Theorem 4.5.2.

5.1. The order of an operator. Let V be a graded commutative algebra. For a linear operator $D : V \rightarrow V$ define a collection,

$$F_n^D : \begin{array}{ccc} \otimes^n V & \longrightarrow & V \\ v_1 \otimes \dots \otimes v_n & \longrightarrow & F_n^D(v_1, \dots, v_n) \end{array}$$

of linear maps by induction: $F_1^D = D$,

$$F_{n+1}^D(v_1, \dots, v_n, v_{n+1}) = F_n^D(v_1, \dots, v_n \cdot v_{n+1}) - F_n^D(v_1, \dots, v_n) \cdot v_{n+1} - (-1)^{|v_n||v_{n+1}|} F_n^D(v_1, \dots, v_{n+1}) \cdot v_n.$$

The operator D is said to have *order* $\leq n$ if $F_{n+1}^D = 0$.

The operators F_n^D are in fact graded symmetric; moreover, if D is a differential in V (that is, $|D| = 1$ and $D^2 = 0$), then the collection, $\{F_n^D : \odot^n V \rightarrow V\}_{n \geq 1}$ defines a $\mathcal{L}ie_\infty$ -structure on the space $V[-1]$ (see [Kr]). Indeed, consider a graded Lie algebra,

$$\text{CoDer}(\odot^{\bullet \geq 1} V) \cong \prod_{n \geq 1} \text{Hom}_{\mathbb{K}}(\odot^n V, V),$$

of coderivations of the graded co-commutative coalgebra $\odot^{\bullet \geq 1} V$. As the differential $D : V \rightarrow V$ is a Maurer-Cartan element in this Lie algebra and the multiplication $\mu : \odot^2 V \rightarrow V$ is its degree zero element, we can gauge transform D ,

$$D \longrightarrow F^D := e^{-\mu} D e^\mu = \sum_{n=0}^{\infty} \frac{1}{n!} [\dots [[D, \mu], \mu], \dots, \mu],$$

into a less trivial Maurer-Cartan element which makes $V[-1]$ into a (less trivial) $\mathcal{L}ie_\infty$ -algebra. It is easy to see that the associated components of the codifferential F^D ,

$$F^D = \left\{ F_{n+1}^D = \frac{1}{n!} [\dots [[D, \mu], \mu], \dots, \mu] : \odot^{n+1} V \rightarrow V \right\}_{n \geq 0}$$

coincide precisely with the defined above tensors F_{n+1}^D which measure a failure of D to respect the multiplication operation in V .

5.2. Batalin-Vilkovisky algebras. A *Batalin-Vilkovisky algebra* is, by definition, a graded commutative algebra V equipped with a degree -1 operator $\Delta : V \rightarrow V$ of order ≤ 2 such that $\Delta^2 = 0$. Denote by \mathcal{BV} the operad whose representations are Batalin-Vilkovisky algebras. This is, therefore, a graded operad generated by corollas,

$$\textcircled{1} \quad \text{and} \quad \begin{array}{c} | \\ \circ \\ / \quad \backslash \\ 1 \quad 2 \end{array} = \begin{array}{c} | \\ \circ \\ \backslash \quad / \\ 2 \quad 1 \end{array}$$

of homological degrees -1 and 0 respectively, modulo the following relations,

$$\begin{array}{c} \textcircled{1} \\ | \\ \textcircled{1} \end{array} = 0, \quad \begin{array}{c} | \\ \circ \\ / \quad \backslash \\ 1 \quad 3 \end{array} = \begin{array}{c} | \\ \circ \\ \backslash \quad / \\ 2 \quad 3 \end{array}, \quad \begin{array}{c} | \\ \textcircled{1} \\ | \\ \circ \\ / \quad \backslash \\ 1 \quad 2 \end{array} = \sum_{n=0}^2 \begin{array}{c} | \\ \circ \\ / \quad \backslash \\ \zeta^n(3) \end{array} - \sum_{n=0}^2 \begin{array}{c} | \\ \circ \\ \backslash \quad / \\ \zeta^n(2) \end{array}$$

where ζ is the cyclic permutation (123) . A nice non-minimal cofibrant resolution of the operad \mathcal{BV} has been constructed in [GTV]. We denote this resolution by \mathcal{BV}_∞^K in this paper, K standing for *Koszul*. The minimal resolution, \mathcal{BV}_∞ , has been constructed in [DCV].

5.3. An operad $\mathcal{BV}_\infty^{com}$. A $\mathcal{BV}_\infty^{com}$ -algebra is, by definition [Kr], a dg graded commutative algebra (V, d) equipped with a countable collections of linear homogeneous maps, $\{\Delta_a : V \rightarrow V, |\Delta_a| = 1 - 2a\}_{k \geq 1}$, such that each Δ_a is of order $\leq a + 1$ and the equations,

$$(41) \quad \sum_{a=0}^n \Delta_a \circ \Delta_{n-a} = 0,$$

hold for any $n \in \mathbb{N}$, where $\Delta_0 := -d$.

Then we compute

$$\begin{aligned}
\rho\left(\delta \begin{array}{c} \circ \\ | \\ \circ \end{array}\right) &= \sum_{\substack{a=b+c \\ b,c \geq 1}} \rho\left(\begin{array}{c} \circ \\ | \\ \circ \end{array}\right) = \sum_{\substack{a=b+c \\ b,c \geq 1}} \sum_{\substack{b+1=p+k \\ k \geq 1, p \geq 0}} \sum_{\substack{c+1=q+l \\ k \geq 1, q \geq 0}} \rho\left(\begin{array}{c} p \\ | \\ k \\ | \\ 1 \end{array}\right) \circ_1 \rho\left(\begin{array}{c} q \\ | \\ l \\ | \\ 1 \end{array}\right) \\
&= \sum_{\substack{a=b+c \\ b,c \geq 1}} \sum_{\substack{b+1=p+k \\ k,p \geq 1}} \sum_{\substack{c+1=q+l \\ k,p \geq 1}} \sum_{k=k'+k''} \begin{array}{c} p \\ | \\ k' \\ | \\ l \\ | \\ k'' \\ | \\ 1 \end{array} = \sum_{\substack{a=p+q+k'+k''+l-2 \\ p,q,k',k'' \geq 0, k'+k'', l \geq 1, \\ k'+k''+p \geq 2, l+q \geq 2}} \begin{array}{c} p \\ | \\ k' \\ | \\ q \\ | \\ l \\ | \\ k'' \\ | \\ 1 \end{array}
\end{aligned}$$

On the other hand

$$\delta\rho\left(\begin{array}{c} \circ \\ | \\ \circ \end{array}\right) = \sum_{\substack{a+1=p+k \\ k \geq 1, p \geq 0}} \delta \begin{array}{c} p \\ | \\ k \\ | \\ 1 \end{array} = \sum \begin{array}{c} p' \\ | \\ l \\ | \\ k'' \\ | \\ k' \\ | \\ 1 \end{array} + \sum \begin{array}{c} p \\ | \\ k' \\ | \\ l \\ | \\ k'' \\ | \\ 1 \end{array} + \begin{array}{c} \circ \\ | \\ p \\ | \\ k \\ | \\ 1 \end{array} + \begin{array}{c} \circ \\ | \\ p \\ | \\ k \\ | \\ 1 \end{array}$$

Notice that the three terms on the right just kill those terms that were excluded before due to the restrictions $k' + k'' + p \geq 2, l + q \geq 2$. \square

APPENDIX A. PROOF OF PROPOSITION 2.7.1

In this section we show that the quadratic algebra \mathcal{A}_n of section 2.7 is Koszul. In fact, we will show the equivalent statement that the Koszul dual algebra $B_n = \mathcal{A}_n^!$ is Koszul. Concretely, B_n is generated by x_1, \dots, x_n with relations $x_i x_j = 0$ for $|i - j| \neq 1$ and $x_i x_{i+1} = -x_{i+1} x_i$.

We denote by $C_n \otimes_{\kappa} B_n$ the Koszul complex of B_n , i. e., the complex $(C_n \otimes B_n, d_{\kappa})$, where C_n is the coalgebra with quadratic corelations $R = \text{span}(\{x_i x_j \mid |i - j| \neq 1\} \cup \{x_i x_{i+1} + x_{i+1} x_i\})$ and $\kappa : C_n \rightarrow B_n$ is the degree -1 map that is zero everywhere except on V , where it identifies $V \subset C_n$ with $V \subset B_n$.

Notice that B_n and C_n are weight graded and the weight k component of B_n , $B_n^{(k)}$ is zero if $k \geq 3$.

The result will be shown by constructing a contracting homotopy h . Let us denote $x_{i_1} \dots x_{i_l} \otimes [x_j x_{j+1}]$ by x and let us define $h : V^{\otimes l} \otimes B_n^{(2)} \rightarrow V^{\otimes l+1} \otimes B_n^{(1)}$, for $n \geq 1$ by

$$\begin{cases} h(x) = x_{i_1} \dots x_{i_l} x_j \otimes x_{j+1} & \text{if } |j - i_l| \neq 1 \\ h(x) = x_{i_1} \dots x_{i_{l-1}} (x_{i_l} x_j + x_j x_{i_l}) \otimes x_{j+1} & \text{if } |j - i_l| = 1 \text{ and } |j - i_{l-1}| \neq 1 \\ h(x) = -x_{i_1} \dots x_{i_l} x_{j+1} \otimes x_j & \text{if } |j - i_l| = 1 \text{ and } |j - i_{l-1}| = 1 \end{cases}$$

A.0.2. Lemma. h maps $RV^{\otimes l-2} \otimes B_n^{(2)}$ to $RV^{\otimes l-1} \otimes B_n^{(1)} \cap V^{\otimes l-1} R \otimes B_n^{(1)}$ and it maps $V^{\otimes a} RV^{\otimes b} \otimes B_n^{(2)}$ to $V^{\otimes a} RV^{\otimes b+1} \otimes B_n^{(1)}$. As a consequence h restricts to a function $C_n^{(l)} \otimes B_n^{(2)} \rightarrow C_n^{(l+1)} \otimes B_n^{(1)}$ and moreover $d_{\kappa} h = \text{id}_{C_n^{(l)} \otimes B_n^{(2)}}$.

Proof. For the first part of the lemma notice that $RV^{\otimes l-2}$ is generated by the elements $x_{i_1} x_{i_2} \dots x_{i_l}$ such that $|i_1 - i_2| \neq 1$ and the elements $(x_{i+1} x_i + x_i x_{i+1}) x_{i_3} \dots x_{i_l}$ ($l \geq 2$).

If we suppose l to be at least 3, it is clear that h does not do anything to the initial two terms, keeping them in R (thus their image will be in $RV^{\otimes l-1} \otimes B_n^{(1)}$) and by the construction of h it is also clear that the image of a generator is in $V^{\otimes l-1} R \otimes B_n^{(1)}$.

In fact, the same ‘‘not touching the R part’’ argument also shows that h maps $V^{\otimes a} RV^{\otimes b}$ to $V^{\otimes a} RV^{\otimes b+1}$ if b is at least 1.

It remains to show this for the case where $b = 0$. In this case, $V^{\otimes l-2} R$ has two classes of generators: $x_{i_1} \dots x_{i_{l-1}} x_{i_l} \otimes [x_j x_{j+1}]$ where $|i_l - i_{l-1}| \neq 1$ and $x_{i_1} \dots x_{i_{l-2}} (x_{i_{l-1}} x_{i_l} + x_{i_l} x_{i_{l-1}}) \otimes [x_j x_{j+1}]$.

For the first class of generators, if $|j - i_n| \neq 1$ the result is clear. If $|j - i_n| = 1$ and $|j - i_{l-1}| \neq 1$ we use the second formula from the definition of h and since $x_{i_{l-1}} x_{i_l} \in R$ and $x_{i_{l-1}} x_j \in R$ the result is also clear. If $|j - i_n| = 1$ and $|j - i_{l-1}| = 1$ we use the third formula and the elements in R are not touched.

For the second class of generators, this verification must be split into some cases.

Let us consider $x = x_{i_1} \dots x_{i_{l-2}}(x_i x_{i+1} + x_{i+1} x_i) \otimes [x_j x_{j+1}]$. To simplify the verification of the calculations, notice that for these generators the third formula of the definition of h will never be used. If $j < i - 1$ or $j > i + 2$ then h only moves x_j before the tensor product and thus $h(x) \in V^{n-2}RV \otimes B_n^{(1)}$.

If $j = i - 1$,

$$h(x) = x_{i_1} \dots x_{i_{l-2}} x_i x_{i+1} x_{i-1} \otimes x_i + x_{i_1} \dots x_{i_{l-2}} x_{i+1} (x_i x_{i-1} + x_{i-1} x_i) \otimes x_i \in V^{\otimes l-2}RV \otimes B_n^{(1)}$$

If $j = i$,

$$\begin{aligned} h(x) &= x_{i_1} \dots x_{i_{l-2}} x_i (x_{i+1} x_i + x_i x_{i+1}) \otimes x_{i+1} + x_{i_1} \dots x_{i_{l-2}} x_{i+1} x_i x_i \otimes x_{i+1} \\ &= x_{i_1} \dots x_{i_{l-2}} (x_i x_{i+1} + x_{i+1} x_i) x_i \otimes x_{i+1} + x_{i_1} \dots x_{i_{l-2}} x_i x_i x_{i+1} \otimes x_{i+1} \in V^{\otimes l-2}RV \otimes B_n^{(1)} \end{aligned}$$

If $j = i + 1$,

$$\begin{aligned} h(x) &= x_{i_1} \dots x_{i_{l-2}} x_i x_{i+1} x_{i+1} \otimes x_{i+2} + x_{i_1} \dots x_{i_{l-2}} x_{i+1} (x_i x_{i+1} + x_{i+1} x_i) \otimes x_{i+2} \\ &= x_{i_1} \dots x_{i_{l-2}} (x_i x_{i+1} + x_{i+1} x_i) x_{i+1} \otimes x_{i+2} + x_{i_1} \dots x_{i_{l-2}} x_{i+1} x_{i+1} x_i \otimes x_{i+2} \in V^{\otimes l-2}RV \otimes B_n^{(1)} \end{aligned}$$

If $j = i + 2$,

$$h(x) = x_{i_1} \dots x_{i_{l-2}} x_i (x_{i+1} x_{i+2} + x_{i+2} x_{i+1}) \otimes x_{i+3} + x_{i_1} \dots x_{i_{l-2}} x_{i+1} x_i x_{i+2} \otimes x_{i+3} \in V^{\otimes l-2}RV \otimes B_n^{(1)}$$

□

We define h on $V^{\otimes 0} \otimes B_n^{(2)} \rightarrow V^{\otimes 1} \otimes B_n^{(1)}$ by $h(1 \otimes [x_j x_{j+1}]) = x_j \otimes x_{j+1}$.

To define $h : C_n^{(l)} \otimes B_n^{(1)} \rightarrow C_n^{(n+1)} \otimes B_n^{(0)} = C^{(l+1)}$, as before we define it on $V^{\otimes l} \otimes B_n^{(1)}$ and we verify that it restricts properly.

Let us denote $x_{i_1} \dots x_{i_{l-1}} x_{i_l} \otimes x_j$ by x and let us define $h : V^{\otimes l} \otimes B_n^{(1)} \rightarrow V^{\otimes l+1}$, by $h(x) =$

$$\begin{cases} x_{i_1} \dots x_{i_l} x_j & \text{if } |j - i_l| \neq 1 \\ 0 & \text{if } j = i_l + 1 \text{ and } |i_l - i_{l-1}| \neq 1 \\ -x_{i_1} \dots x_{i_{l-2}} x_{i_l} x_{i_{l-1}} x_j & \text{if } j = i_l + 1, |i_l - i_{l-1}| = 1, |i_l - i_{l-2}| \neq 1 \\ x_{i_1} \dots x_{i_{l-2}} x_{i_{l-1}} (x_{i_l} x_j + x_j x_{i_l}) & \text{if } j = i_l + 1, |i_l - i_{l-1}| = 1, |i_l - i_{l-2}| = 1 \\ x_{i_1} \dots x_{i_{l-1}} (x_{i_l} x_j + x_j x_{i_l}) & \text{if } j = i_l - 1 \text{ and } |j - i_{l-1}| \neq 1 \\ x_{i_1} \dots x_{i_{l-2}} (x_{i_{l-1}} x_j + x_j x_{i_{l-1}}) x_{i_l} + x_{i_1} \dots x_{i_l} x_j & \text{if } j = i_l - 1, |j - i_{l-1}| = 1, |j - i_{l-2}| \neq 1 \\ 0 & \text{if } j = i_l - 1, |j - i_{l-1}| = 1, |j - i_{l-2}| = 1 \end{cases}$$

Interpret this definition for $l \leq 2$ in the following way: Whenever i_{l-2} or i_{l-1} are not defined, take the case in the definition where the absolute value of the difference is different from 1.

Notice that the image of h sits inside $V^{\otimes l-1}R$.

A.0.3. Lemma. h maps $V^{\otimes a}RV^{\otimes b} \otimes B_n^{(1)}$ to $V^{\otimes a}RV^{\otimes b+1}$. Therefore h restricts to a function $C_n^{(l)} \otimes B_n^{(1)} \rightarrow C_n^{(l+1)}$. Moreover, $dh + hd = id$.

Proof. As before, the R part in $V^{\otimes a}RV^{\otimes b}$ is not touched by h if b is at least 2 hence $V^{\otimes a}RV^{\otimes b} \otimes B_n^{(1)}$ is sent to $V^{\otimes a}RV^{\otimes b+1}$.

If $b = 1$ the same argument holds, except for the 6th case of the definition of h , when $j = i_l - 1$, $|j - i_{l-1}| = 1$ and $|j - i_{l-2}| \neq 1$. The only way to use this case of the definition of h is when we apply h to the element

$$x = x_{i_1} \dots x_{i_{l-3}} (x_i x_{i+1} + x_{i+1} x_i) x_{j+1} \otimes x_j$$

and either $|j - i| = 1$ (which implies $|j - (i + 1)| \neq 1$) or $|j - (i + 1)| = 1$ (which implies $|j - i| \neq 1$).

If $|j - i| = 1$,

$$\begin{aligned} h(x) &= x_{i_1} \dots x_{i_{l-3}} [x_i x_{i+1} (x_{j+1} x_j + x_j x_{j+1}) + x_{i+1} (x_i x_j + x_j x_i) x_{j+1} + x_{i+1} x_i x_{j+1} x_j] \\ &= x_{i_1} \dots x_{i_{l-3}} [(x_i x_{i+1} + x_{i+1} x_i) x_j x_{j+1} + (x_i x_{i+1} + x_{i+1} x_i) x_{j+1} x_j + x_{i+1} x_j x_i x_{j+1}] \in V^{\otimes a}RV^{\otimes 2} \end{aligned}$$

If $|j - (i + 1)| = 1$,

$$\begin{aligned} h(x) &= x_{i_1} \dots x_{i_{l-3}} [x_i(x_{i+1}x_j + x_jx_{i+1})x_{j+1} + x_ix_{i+1}x_{j+1}x_j + x_{i+1}x_i(x_{j+1}x_j + x_jx_{j+1})] \\ &= x_{i_1} \dots x_{i_{l-3}} [(x_ix_{i+1} + x_{i+1}x_i)x_jx_{j+1} + (x_ix_{i+1} + x_{i+1}x_i)x_{j+1}x_j] + x_ix_jx_{i+1}x_{j+1} \in V^{\otimes a}RV^{\otimes 2} \end{aligned}$$

Let us show that $V^{\otimes l-2}R$ is sent to $V^{\otimes l-2}RV$ by looking at each generator and meanwhile we will check that $dh + hd = id$. $V^{\otimes l-2}R$ has two types of generators: $x_{i_1} \dots x_{i_{l-1}}x_{i_l} \otimes x_j$ where $|i_l - i_{l-1}| \neq 1$ and $x_{i_1} \dots x_{i_{l-2}}(x_ix_{i+1} + x_{i+1}x_i) \otimes x_j$.

For the first type of generators, let us denote $x_{i_1} \dots x_{i_{l-1}}x_{i_l} \otimes x_j$ by x . If $|j - i_l| \neq 1$, $h(x) = x_{i_1} \dots x_{i_l}x_j$ clearly belongs to $V^{\otimes l-2}RV$ and since $dx = 0$ we have that $(dh + hd)(x) = x$.

If $j = i_l + 1$, $h(x) = 0$ and $(dh + hd)(x) = x$.

If $j = i_l - 1$ and $|j - i_{l-1}| \neq 1$,

$$h(x) = x_{i_1} \dots x_{i_{l-1}}(x_{i_l}x_j + x_jx_{i_l}) \in V^{\otimes l-2}RV.$$

$$\begin{aligned} hd(x) &= h(-x_{i_1} \dots x_{i_{l-1}} \otimes x_jx_{i_l}) = -x_{i_1} \dots x_{i_{l-1}}x_j \otimes x_{i_l} \text{ and thus} \\ (hd + dh)(x) &= x. \end{aligned}$$

If $j = i_l - 1$, $|j - i_{l-1}| = 1$ and $|j - i_{l-2}| \neq 1$

$$h(x) = x_{i_1} \dots x_{i_{l-2}}(x_{i_{l-1}}x_j + x_jx_{i_{l-1}})x_{i_l} + x_{i_1} \dots x_{i_{l-1}}x_{i_l}x_j \in V^{\otimes l-2}RV.$$

$$hd(x) = h(-x_{i_1} \dots x_{i_{l-1}} \otimes x_jx_{i_l}) = -x_{i_1} \dots x_{i_{l-2}}(x_{i_{l-1}}x_j + x_jx_{i_{l-1}}) \otimes x_{i_l} \Rightarrow (hd + dh)(x) = x.$$

If $j = i_l - 1$, $|j - i_{l-1}| = 1$ and $|j - i_{l-2}| = 1$, $h(x) = 0$

$$hd(x) = h(-x_{i_1} \dots x_{i_{l-1}} \otimes x_jx_{i_l}) = -(-x) = x$$

$$\Rightarrow (hd + dh)(x) = x.$$

It only remains to check generators of the second type, i. e., elements $x := x_{i_1} \dots x_{i_{l-2}}(x_ix_{i+1} + x_{i+1}x_i) \otimes x_j$. If $j < i - 1$ or $j > i + 2$ it is clear that $h(x) \in V^{\otimes l-2}RV$ and $(dh + hd)(x) = dh(x) = x$.

If $j = i - 1$,

$$h(x) = x_{i_1} \dots x_{i_{l-2}}x_ix_{i+1}x_{i-1} + x_{i_1} \dots x_{i_{l-2}}x_{i+1}(x_ix_{i-1} + x_{i-1}x_i) \in V^{\otimes l-2}RV$$

$$\begin{aligned} hd(x) &= h(x_{i_1} \dots x_{i_{l-2}}x_{i+1} \otimes x_ix_{i-1}) = -x_{i_1} \dots x_{i_{l-2}}x_{i+1}x_{i-1} \otimes x_{i+1} \\ \Rightarrow (dh + hd)(x) &= x. \end{aligned}$$

If $j = i$,

$$h(x) = x_{i_1} \dots x_{i_{l-2}}x_i(x_{i+1}x_i + x_ix_{i+1}) + x_{i_1} \dots x_{i_{l-2}}x_{i+1}x_ix_i \in V^{\otimes l-2}RV$$

$$\begin{aligned} hd(x) &= h(x_{i_1} \dots x_{i_{l-2}}x_i \otimes x_{i+1}x_i) = -x_{i_1} \dots x_{i_{l-2}}x_ix_i \otimes x_{i+1} \\ \Rightarrow (dh + hd)(x) &= x. \end{aligned}$$

If $j = i + 1$ and $|i - i_{l-2}| \neq 1$,

$$h(x) = x_{i_1} \dots x_{i_{l-2}}x_ix_{i+1}x_{i+1} - x_{i_1} \dots x_{i_{l-2}}x_ix_{i+1}x_{i+1} = 0$$

$$hd(x) = h(x_{i_1} \dots x_{i_{l-2}}x_{i+1} \otimes x_ix_{i+1}) = x_{i_1} \dots x_{i_{l-2}}(x_{i+1}x_i + x_ix_{i+1}) \otimes x_{i+1} = x$$

If $j = i + 1$ and $|i - i_{l-2}| = 1$,

$$h(x) = x_{i_1} \dots x_{i_{l-2}} x_i x_{i+1} x_{i+1} + x_{i_1} \dots x_{i_{l-2}} x_{i+1} (x_i x_{i+1} + x_{i+1} x_i) \in V^{\otimes l-2} RV$$

$$\begin{aligned} hd(x) &= h(x_{i_1} \dots x_{i_{l-2}} x_{i+1} \otimes x_i x_{i+1}) = -x_{i_1} \dots x_{i_{l-2}} x_{i+1} x_{i+1} \otimes x_i. \\ \Rightarrow (dh + hd)(x) &= x. \end{aligned}$$

If $j = i + 2$ and $|i_{l-2} - (i + 1)| \neq 1$

$$h(x) = -x_{i_1} \dots x_{i_{l-2}} x_{i+1} x_i x_{i+2} + x_{i_1} \dots x_{i_{l-2}} x_{i+1} x_i x_{i+2} = 0$$

$$\begin{aligned} hd(x) &= h(x_{i_1} \dots x_{i_{l-2}} x_i \otimes x_{i+1} x_{i+2}) = x_{i_1} \dots x_{i_{l-2}} (x_i x_{i+1} + x_{i+1} x_i) \otimes x_{i+2} \\ \Rightarrow (dh + hd)(x) &= x. \end{aligned}$$

If $j = i + 2$ and $|i_{l-2} - (i + 1)| = 1$

$$h(x) = x_{i_1} \dots x_{i_{l-2}} x_i (x_{i+1} x_{i+2} + x_{i+2} x_{i+1}) + x_{i_1} \dots x_{i_{l-2}} x_{i+1} x_i x_{i+2} \in V^{\otimes l-2} RV$$

$$\begin{aligned} hd(x) &= h(x_{i_1} \dots x_{i_{l-2}} x_i \otimes x_{i+1} x_{i+2}) = -x_{i_1} \dots x_{i_{l-2}} x_i x_{i+2} \otimes x_{i+1} \\ \Rightarrow (dh + hd)(x) &= x. \end{aligned} \quad \square$$

APPENDIX B. COMPUTATIONS OF THE COHOMOLOGY OF DEFORMATION COMPLEXES

In this section we compute the cohomology of several of the deformation complexes and show theorems 4.5.2 and 4.9.2.

B.1. The proof of Theorem 4.5.2. Let us recall the definition of the graph complex $\widehat{\mathcal{GC}}_3^{or}$ from [W3, section 3.3]. The elements of $\widehat{\mathcal{GC}}_3^{or}$ are \mathbb{K} -linear series in directed acyclic graphs with outgoing legs such that all vertices are at least bivalent, and such that there are no bivalent vertices with one incoming and one outgoing edge.⁶ We set to zero graphs containing vertices without outgoing edges. Here is an example graph:



The degrees are computed just as for graphs occurring in \mathcal{GC}_3^{or} , with the external legs considered to be of degree 0. For the description of the differential we refer the reader to [W3, section 3.3].

There is a map $\Psi : \mathcal{GC}_3^{or} \rightarrow \widehat{\mathcal{GC}}_3^{or}$ sending a graph Γ to the linear combination

$$(42) \quad \Gamma \mapsto \sum_{j=1}^{\infty} \underbrace{\Gamma}_{j \times}$$

where the picture on the right means that one should sum over all ways of connecting j outgoing edges to the graph Γ . Graphs for which there remain vertices with no outgoing edge are identified with 0.

The following proposition has been shown in loc. cit.

B.1.1. Proposition (Proposition 3 of [W3]). *The map $\Psi : \mathcal{GC}_3^{or} \rightarrow \widehat{\mathcal{GC}}_3^{or}$ is a quasi-isomorphism up to the class in $H(\widehat{\mathcal{GC}}_n^{or})$ represented by the graph cocycle*

$$(43) \quad \sum_{j \geq 2} (j-1) \underbrace{\Gamma}_{j \times}$$

There is a map $G : \widehat{\mathcal{GC}}_3^{or} \rightarrow \text{Der}(\mathcal{LieB}_\infty)$ sending a graph Γ in \mathcal{GC}_3^{or} to the series

$$(44) \quad G(\Gamma) = \sum \underbrace{\Gamma}_{n \times}$$

⁶The last condition is again not present on [W3], but it does not change the cohomology.

The map $F : \mathcal{GC}_3^{or} \rightarrow \text{Der}(\mathcal{Lie}\mathcal{B}_\infty)$ from Theorem 4.5.2 factors through the map G above, i. e., it can be written as the composition

$$\mathcal{GC}_3^{or} \xrightarrow{\Psi} \widehat{\mathcal{GC}}_3^{or} \xrightarrow{G} \text{Der}(\mathcal{Lie}\mathcal{B}_\infty).$$

In view of Proposition B.1.1 Theorem 4.5.2 hence follows immediately from the following result.

B.1.2. Proposition. *The map $G : \widehat{\mathcal{GC}}_3^{or} \rightarrow \text{Der}(\mathcal{Lie}\mathcal{B}_\infty)$ is a quasi-isomorphism.*

Proof. For a graph in $\text{Der}(\mathcal{Lie}\mathcal{B}_\infty)$ we will call its *skeleton* the graph obtained in the following way:

- (1) Remove all input legs and recursively remove all valence 1 vertices created.
- (2) Remove valence 2 vertices with one incoming and one outgoing edge and connect the two edges.

An example of a graph and its skeleton the following



We put a filtration on $\text{Der}(\mathcal{Lie}\mathcal{B}_\infty)$ by the total number of vertices in the skeleton. Let $\text{grDer}(\mathcal{Lie}\mathcal{B}_\infty)$ be the associated graded. Note that for elements in the image of some graph $\Gamma \in \widehat{\mathcal{GC}}_3^{or}$ under G the skeleton is just the graph Γ , and hence there is a map of complexes $\widehat{\mathcal{GC}}_3^{or} \rightarrow \text{grDer}(\mathcal{Lie}\mathcal{B}_\infty)$, where we consider the left hand side with zero differential. We claim that the induced map $\widehat{\mathcal{GC}}_3^{or} \rightarrow H(\text{grDer}(\mathcal{Lie}\mathcal{B}_\infty))$ is an isomorphism. From this claim the Proposition follows immediately by a standard spectral sequence argument.

The differential on $\text{grDer}(\mathcal{Lie}\mathcal{B}_\infty)$ does not change the skeleton. Hence the complex $\text{grDer}(\mathcal{Lie}\mathcal{B}_\infty)$ splits into a direct product of complexes, say \tilde{C}_γ , one for each skeleton γ

$$\text{grDer}(\mathcal{Lie}\mathcal{B}_\infty) = \prod_{\gamma} \tilde{C}_\gamma.$$

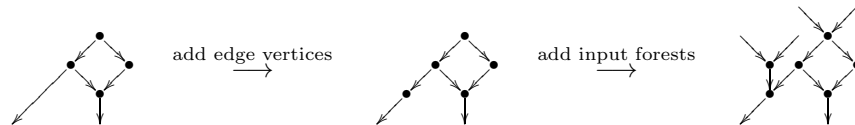
Furthermore, each skeleton represents an automorphism class of graphs, and we may write

$$\tilde{C}_\gamma = C_{\tilde{\gamma}}^{\text{Aut}_\gamma}$$

where $C_{\tilde{\gamma}}$ is an appropriately defined complex for one representative $\tilde{\gamma}$ of the isomorphism class γ and Aut_γ is the automorphism group associated to the skeleton. In other words, the $\tilde{\gamma}$ now has distinguishable vertices and edges. More concretely, the complex $C_{\tilde{\gamma}}$ is the complex of \mathbb{K} -linear series of graphs obtained from $\tilde{\gamma}$ by

- (1) Adding some bivalent vertices with one input and one output on edges. We call these vertices “edge vertices”.
- (2) Attaching input forests at the vertices, such that all vertices are at least trivalent and have at least one input and one output. We call the forest attached to a vertex the forest of that vertex.

An example is the following:



We next put another filtration on $C_{\tilde{\gamma}}$ by the number of edge vertices added in the first step above and consider the associated graded $\text{gr}C_{\tilde{\gamma}}$. Note that the differential $\text{gr}C_{\tilde{\gamma}}$ acts on each of the forests attached to the vertices separately and hence the complex splits into a (completed) tensor product of complexes, one for each such vertex. Let us call the complex made from the possible forests at the vertex v the forest complex at that vertex. By the same argument showing that the cohomology of a free Lie_∞ algebra generated by a single generator is two dimensional, we find that the forest complex at v has either one or two dimensional cohomology. If vertex v has no incoming edge that it is one dimensional, the class being represented by the forests

$$\sum_{j \geq 1} \overbrace{\left(\begin{array}{c} j \times \\ \vdots \\ v \end{array} \right)}.$$

If the vertex v already has an incoming edge, then there is one additional class obtained by not adding any input forest.

Hence we find that $H(\text{gr}C_{\tilde{\gamma}})$ is spanned by graphs obtained from $\tilde{\gamma}$ as follows:

- (1) Add some bivalent vertices with one input and one output on edges.
- (2) For each vertex that is either not at least trivalent or does not have an incoming edge, sum over all ways of attaching incoming legs at that vertex.
- (3) For at least trivalent vertices with an incoming edge, there is a choice of either not adding anything at that vertex, or summing over all ways of attaching incoming legs at that vertex. Let us call the vertices for which the first choice is made bald vertices and the others hairy.

Let us look at the next page in the spectral sequence associated to our filtration on $C_{\tilde{\gamma}}$. The differential creates one edge vertex by either splitting an existing edge vertex or by splitting a skeleton vertex. Again, the complex splits into a product of complexes, one for each edge of $\tilde{\gamma}$. For each such edge we have to consider 3 cases separately:

- (1) Both endpoints in $\tilde{\gamma}$ are hairy.
- (2) Both endpoints in $\tilde{\gamma}$ are bald.
- (3) One endpoints is hairy and one is bald.

We leave it to the reader to check that:

- (1) In the first case the cohomology is one-dimensional, represented by a single edge without edge vertices.
- (2) In the third case the cohomology vanishes.

Since there is necessarily at least one hairy vertex in the graph, the second assertion implies that if there is a bald vertex as well, the resulting complex is acyclic. Hence all vertices must be hairy. By the first assertion the cohomology is one-dimensional for each skeleton. One easily checks that the representative is exactly the image of the skeleton considered as element in $\widehat{\text{GC}}_3^{or}$. Hence the proposition follows. \square

B.1.3. Remark. There is also an alternative way to compute the cohomology of the deformation complex $\text{Der}(\mathcal{L}ie\mathcal{B}_\infty)$. Namely, by Koszulness of $\mathcal{L}ie\mathcal{B}$ this complex is quasi-isomorphic to $\text{Def}(\mathcal{L}ie\mathcal{B}_\infty \rightarrow \mathcal{L}ie\mathcal{B})[1]$. It is well known that the prop governing Lie bialgebras $\mathcal{L}ie\mathcal{B}P$ may be written as

$$\mathcal{L}ie\mathcal{B}P(n, m) \cong \bigoplus_N \mathcal{L}ieP(n, N) \otimes_{\mathbb{S}_N} co\mathcal{L}ieP(N, m)$$

using the props governing Lie algebras and Lie coalgebras. Interpreting elements of the above prop as linear combinations directed acyclic graphs, the sub-properad $\mathcal{L}ie\mathcal{B}$ may be obtained as that formed by the connected such graphs. It is hence an easy exercise to check that $\text{Def}(\mathcal{L}ie\mathcal{B}_\infty \rightarrow \mathcal{L}ie\mathcal{B})[1]$ is identical to the complex $\text{Def}(\text{hoe}_2 \rightarrow e_2)_{\text{conn}}[1]$ from [W1], up to unimportant completion issues. The cohomology of the latter complex has been computed in loc. cit. to be

$$H(\text{GC}_2) \oplus \bigoplus_{j=1,5,\dots} \mathbb{K}[2-j] \oplus \mathbb{K}.$$

Using the main result of [W3] this agrees with the cohomology as computed by Theorem 4.5.2. Conversely, the above proof of Theorem 4.5.2 together with this remark yields an alternative proof of the main result of [W3].

B.2. The proof of Theorem 4.9.2. Let us next consider Theorem 4.9.2, whose proof will be a close analog of that of Theorem 4.5.2 in the previous subsection. There is a natural differential graded Lie algebra structure on $\widehat{\text{GC}}_3^{or}$ such that the map $\Psi : \text{GC}_3^{or} \rightarrow \widehat{\text{GC}}_3^{or}$ from the previous section is a map of Lie algebras. The map Ψ extends \hbar -linearly to a map of graded Lie algebras $\Psi_\hbar : \text{GC}_3^{or}[[\hbar]] \rightarrow \widehat{\text{GC}}_3^{or}[[\hbar]]$. The Maurer-Cartan element $\Phi_\hbar \in \text{GC}_3^{or}[[\hbar]]$ from Proposition 4.8 is sent to a Maurer-Cartan element $\hat{\Phi}_\hbar := \Psi_\hbar(\Phi_\hbar) \in \widehat{\text{GC}}_3^{or}[[\hbar]]$. We endow $\widehat{\text{GC}}_3^{or}[[\hbar]]$ with the differential

$$d_\hbar \Gamma = [\hat{\Phi}_\hbar, \Gamma].$$


In particular it follows that we have a map of differential graded Lie algebras

$$\Psi_{\hbar} : (\mathrm{GC}_3^{or}[[\hbar]], d_{\hbar}) \rightarrow (\widehat{\mathrm{GC}}_3^{or}[[\hbar]], d_{\hbar}).$$

The map F_{\hbar} from Theorem 4.9.2 factors through $\widehat{\mathrm{GC}}_3^{or}[[\hbar]]$:

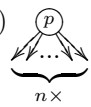
$$\mathrm{GC}_3^{or}[[\hbar]] \xrightarrow{\Psi_{\hbar}} \widehat{\mathrm{GC}}_3^{or}[[\hbar]] \xrightarrow{G_{\hbar}} \mathrm{Der}(\mathcal{L}ie^{\circ}\mathcal{B}_{\infty}).$$

The second map $G_{\hbar} : \widehat{\mathrm{GC}}_3^{or}[[\hbar]] \rightarrow \mathrm{Der}(\mathcal{L}ie^{\circ}\mathcal{B}_{\infty})$ sends $\hbar^N \Gamma$, for $\Gamma \in \widehat{\mathrm{GC}}_3^{or}$ to

$$\sum_{j \geq 1} \sum_{k=0}^N \hbar^{N-k} \Gamma_k$$


where we again sum over all ways of attaching the incoming legs, setting to zero graphs with vertices without incoming edges. Furthermore, Γ_k is the linear combination of graphs obtained by summing over all ways of assigning weights to the vertices of Γ such that the total weight is k . We have the following two results, from which Theorem 4.9.2 immediately follows.

B.2.1. Proposition. *The map $\Psi_{\hbar} : (\mathrm{GC}_3^{or}[[\hbar]], d_{\hbar}) \rightarrow (\widehat{\mathrm{GC}}_3^{or}[[\hbar]], d_{\hbar})$ is a quasi-isomorphism up to the classes $T\mathbb{K}[[\hbar]] \subset \mathrm{Der}(\mathcal{L}ie^{\circ}\mathcal{B}_{\infty})$ where*

$$T = \sum_{n,p} (n + 2p - 2) \Gamma_{n,p}$$


Proof sketch. Take filtrations on $\mathrm{GC}_3^{or}[[\hbar]]$ and $\widehat{\mathrm{GC}}_3^{or}[[\hbar]]$ by the power of \hbar . The differential on the associated graded spaces is the \hbar -linear extension of the differentials on GC_3^{or} and $\widehat{\mathrm{GC}}_3^{or}$. Hence by Proposition B.1.1 the map Ψ_{\hbar} is a quasi-isomorphism on the level of the associated graded spaces, up to the classes above. The result follows by a standard spectral sequence argument, noting that the above element T is indeed d_{\hbar} -closed. \square

B.2.2. Proposition. *The map $G_{\hbar} : \widehat{\mathrm{GC}}_3^{or}[[\hbar]] \rightarrow \mathrm{Der}(\mathcal{L}ie^{\circ}\mathcal{B}_{\infty})$ is a quasi-isomorphism.*

Proof sketch. Take filtrations on $\widehat{\mathrm{GC}}_3^{or}[[\hbar]]$ and

$$\mathrm{Der}(\mathcal{L}ie^{\circ}\mathcal{B}_{\infty}) \cong \prod_{n,m \geq 1} (\mathcal{L}ie^{\circ}\mathcal{B}_{\infty}(n,m) \otimes \mathrm{sgn}_n \otimes \mathrm{sgn}_m)^{\mathbb{S}_n \times \mathbb{S}_m} [[\hbar]] [-n - m + 1]$$

by genus and by powers of \hbar . Then we claim that the induced map on the associated graded complexes $\mathrm{gr}G_{\hbar} : \mathrm{gr}\widehat{\mathrm{GC}}_3^{or}[[\hbar]] \rightarrow \mathrm{gr}\mathrm{Der}(\mathcal{L}ie^{\circ}\mathcal{B}_{\infty})$ is a quasi-isomorphism, thus showing the proposition by a standard spectral sequence argument.

To show the claim, we proceed analogously to the proof of Proposition B.1.2. Let us go through the proof again and highlight only the differences. The skeleton of a graph is defined as before, except that one also forgets the weights of all vertices. The complex $\mathrm{gr}\mathrm{Der}(\mathcal{L}ie^{\circ}\mathcal{B}_{\infty})$ splits into a product of subcomplexes that we again call \tilde{C}_{γ} , one for each skeleton γ . Again

$$\tilde{C}_{\gamma} = C_{\tilde{\gamma}}^{\mathrm{Aut}_{\tilde{\gamma}}}$$

for some representative $\tilde{\gamma}$ of the isomorphism class γ . Hence it again suffices to compute the cohomology of $C_{\tilde{\gamma}}$. Graphs contributing are again obtained by adding edge vertices and input forests, except that now all vertices are also assigned an arbitrary weight. Again we take a filtration on the number of edge vertices, which leaves us with the task of computing the cohomology of a complex associated to one forest attached to a vertex v . We find that representatives of cohomology classes are either:

- Vertex v with any weight and no attached forest. Let us call such a v again bald.
- Vertex v with weight 0 and input legs attached in all possible ways, let us call such a v again hairy.

The differential on the second page of the spectral sequence again adds one edge vertex, which however can have a non-zero weight now, and if it has a non-zero weight it may be bald. We may introduce another filtration by the number of non-hairy edge vertices. The differential on the associated graded creates one hairy edge vertex. The resulting complex is a tensor product of complexes, one for each edge. The complexes associated to each edge again can have three different types: (i) both endpoints in the skeleton are hairy, (ii) both are bald or (iii) one is hairy, one is bald. Again one checks that in case (iii) the complex is acyclic and in case (i) one-dimensional, the cohomology class represented by a single edge. Hence, since at least one vertex must be hairy, all vertices must be. Hence we recover at this stage the image of $\widehat{\mathcal{GC}}_3^{or}[[\hbar]]$ and are done. \square

APPENDIX C. COMPUTATION OF THE COHOMOLOGY OF THE OPERAD $\mathcal{BV}_\infty^{com}$

C.1. An equivalent definition of the operad \mathcal{BV} . Let \mathcal{L}^\diamond be an operad generated by two degree -1 corollas, $\begin{array}{c} \textcircled{1} \\ | \\ \textcircled{1} \end{array}$ and $\begin{array}{c} \textcircled{0} \\ / \quad \backslash \\ 1 \quad 2 \end{array} = \begin{array}{c} \textcircled{0} \\ / \quad \backslash \\ 2 \quad 1 \end{array}$, subject to the following relations,

$$\begin{array}{c} \textcircled{1} \\ | \\ \textcircled{1} \end{array} = 0, \quad \begin{array}{c} \textcircled{1} \\ | \\ \textcircled{0} \\ / \quad \backslash \\ \quad \quad | \\ \quad \quad \textcircled{1} \end{array} + \begin{array}{c} \textcircled{0} \\ / \quad \backslash \\ \textcircled{1} \quad \quad \\ | \quad \quad \quad \\ \textcircled{1} \end{array} + \begin{array}{c} \textcircled{0} \\ / \quad \backslash \\ \quad \quad \textcircled{1} \\ | \quad \quad \quad \\ \textcircled{1} \end{array} = 0, \quad \begin{array}{c} \textcircled{0} \\ / \quad \backslash \\ \textcircled{0} \quad 3 \\ / \quad \backslash \\ 1 \quad 2 \end{array} + \begin{array}{c} \textcircled{0} \\ / \quad \backslash \\ \textcircled{0} \quad 1 \\ / \quad \backslash \\ 2 \quad 3 \end{array} + \begin{array}{c} \textcircled{0} \\ / \quad \backslash \\ \textcircled{0} \quad 2 \\ / \quad \backslash \\ 3 \quad 1 \end{array} = 0$$

Let \mathcal{Com} be the operad of commutative algebras with the generator controlling the graded commutative multiplication denoted by $\begin{array}{c} \textcircled{0} \\ / \quad \backslash \\ \quad \quad | \\ \quad \quad \textcircled{0} \end{array}$. Define an operad, \mathcal{BV} , of Batalin-Vilkovisky algebras as the free operad generated by operads \mathcal{L}^\diamond and \mathcal{Com} modulo the following relations,

$$(45) \quad \begin{array}{c} \textcircled{0} \\ / \quad \backslash \\ 1 \quad 2 \end{array} = \begin{array}{c} \textcircled{1} \\ | \\ \textcircled{0} \\ / \quad \backslash \\ \quad \quad | \\ \quad \quad \textcircled{0} \end{array} - \begin{array}{c} \textcircled{0} \\ / \quad \backslash \\ \textcircled{1} \quad \quad \\ | \quad \quad \quad \\ \textcircled{0} \end{array} - \begin{array}{c} \textcircled{0} \\ / \quad \backslash \\ \quad \quad \textcircled{1} \\ | \quad \quad \quad \\ \textcircled{0} \end{array}, \quad \begin{array}{c} \textcircled{0} \\ / \quad \backslash \\ \textcircled{0} \quad 3 \\ / \quad \backslash \\ 1 \quad 2 \end{array} - \begin{array}{c} \textcircled{0} \\ / \quad \backslash \\ \textcircled{0} \quad 1 \\ / \quad \backslash \\ 2 \quad 3 \end{array} - \begin{array}{c} \textcircled{0} \\ / \quad \backslash \\ \textcircled{0} \quad 2 \\ / \quad \backslash \\ 3 \quad 1 \end{array} = 0.$$

In fact, the second relation in (45) follows from the previous ones. We keep it the list in order to define, following [GTV], an operad $q\mathcal{BV}$ as an operad freely generated by \mathcal{L}^\diamond and \mathcal{Com} modulo a version of relations (45) in which the first relation is replaced by the following one,

$$\begin{array}{c} \textcircled{1} \\ | \\ \textcircled{0} \\ / \quad \backslash \\ \quad \quad | \\ \quad \quad \textcircled{0} \end{array} - \begin{array}{c} \textcircled{0} \\ / \quad \backslash \\ \textcircled{1} \quad \quad \\ | \quad \quad \quad \\ \textcircled{0} \end{array} - \begin{array}{c} \textcircled{0} \\ / \quad \backslash \\ \quad \quad \textcircled{1} \\ | \quad \quad \quad \\ \textcircled{0} \end{array} = 0.$$

Being a quotient of a free operad, the operad \mathcal{BV} inherits an increasing filtration by the number of vertices in the trees. It is clear that there is a morphism,

$$g : q\mathcal{BV} \longrightarrow gr(\mathcal{BV}),$$

from $q\mathcal{BV}$ into the associated graded operad.

C.2. Proposition [GTV]. *The morphism $g : q\mathcal{BV} \longrightarrow gr(\mathcal{BV})$ is an isomorphism.*

C.3. Remark. The relations in the operad $q\mathcal{BV}$ are homogeneous. It is easy to see that, as an \mathbb{S} -module, $q\mathcal{BV}$ is isomorphic to $\mathcal{Com} \circ \mathcal{L}^\diamond$, the vector space spanned by graphs from \mathcal{Com} whose legs are decorated with elements from \mathcal{L}^\diamond .

C.4. An auxiliary dg operad. For any natural number $a \geq 1$ define by induction (over the number, $k = 1, 2, \dots, a + 1$, of input legs) a collection of $a + 1$ elements,

$$(46) \quad \begin{array}{c} \boxed{a} \\ | \\ \text{---} \end{array} := \begin{array}{c} \circ \\ | \\ \text{---} \end{array}, \dots, \begin{array}{c} \boxed{a} \\ | \\ \text{---} \\ / \quad \backslash \\ \dots \quad \dots \\ 1 \quad 2 \quad \dots \quad k-1 \quad k \end{array} := \begin{array}{c} \boxed{a} \\ | \\ \text{---} \\ / \quad \backslash \\ \dots \quad \dots \\ 1 \quad 2 \quad \dots \quad k-2 \quad k-1 \quad k \end{array} - \begin{array}{c} \text{---} \\ / \quad \backslash \\ \dots \quad \dots \\ 1 \quad 2 \quad \dots \quad k-2 \quad k-1 \quad k \end{array} - \begin{array}{c} \text{---} \\ / \quad \backslash \\ \dots \quad \dots \\ 1 \quad 2 \quad \dots \quad k-2 \quad k-1 \quad k \end{array}$$

of the operad $\mathcal{BV}_\infty^{\text{com}}$. If $\rho : \mathcal{BV}_\infty^{\text{com}} \rightarrow \mathcal{E}nd_V$ is a representation, then, in the notation of §5.1,

$$\rho \left(\begin{array}{c} \boxed{a} \\ | \\ \text{---} \\ / \quad \backslash \\ \dots \quad \dots \\ 1 \quad 2 \quad \dots \quad k-1 \quad k \end{array} \right) = F_k^{\Delta_a}.$$

Note that $F_k^{\Delta_a}$ identically vanishes for $k \geq a + 2$ as the operator Δ_a is, by its definition, of order $\leq a + 1$; this is the reason why we defined the above elements of $\mathcal{BV}_\infty^{\text{com}}$ only in the range $1 \leq k \leq a + 1$: for all other k these elements vanish identically due to the relations between the generators of $\mathcal{BV}_\infty^{\text{com}}$.

Consider next a free operad, $\mathcal{L}_\infty^\diamond$, generated, for all integers $p \geq 0, k \geq 1$ with $p + k \geq 2$, by the symmetric corollas

$$\begin{array}{c} \textcircled{p} \\ | \\ \text{---} \\ / \quad \backslash \\ \dots \quad \dots \\ 1 \quad 2 \quad \dots \quad k \end{array} = \begin{array}{c} \textcircled{p} \\ | \\ \text{---} \\ / \quad \backslash \\ \dots \quad \dots \\ \sigma(1) \quad \sigma(2) \quad \dots \quad \sigma(k) \end{array} \quad \forall \sigma \in \mathbb{S}_n,$$

of homological degree $3 - 2k - 2p$, and equipped with the following differential

$$d \begin{array}{c} \textcircled{p} \\ | \\ \text{---} \\ / \quad \backslash \\ \dots \quad \dots \\ 1 \quad 2 \quad \dots \quad k \end{array} = \sum_{\substack{p=q+r \\ [k]=I_1 \sqcup I_2}} \begin{array}{c} \textcircled{r} \\ | \\ \text{---} \\ / \quad \backslash \\ \dots \quad \dots \\ I_2 \end{array} \begin{array}{c} \textcircled{q} \\ | \\ \text{---} \\ / \quad \backslash \\ \dots \quad \dots \\ I_1 \end{array}$$

Representations, $\rho : \mathcal{L}_\infty^\diamond \rightarrow \mathcal{E}nd_V$, of this operad in a dg vector space (V, d) is the same thing as continuous representations of the operad $\mathcal{L}ie_\infty\{1\}[[\hbar]]$ in the topological vector space $V[[\hbar]]$ equipped with the differential

$$d + \sum_{p \geq 1} \hbar^p \Delta_p, \quad \Delta_p := \rho \left(\begin{array}{c} \circ \\ | \\ \text{---} \end{array} \right).$$

where the formal parameter \hbar is assumed to have homological degree 2.

C.4.1. Proposition. *The cohomology of the dg operad $\mathcal{L}_\infty^\diamond$ is the operad \mathcal{L}^\diamond defined in §C.1, i.e. $\mathcal{L}_\infty^\diamond$ is a minimal resolution of \mathcal{L}^\diamond .*

Proof. The dg operad $\mathcal{L}_\infty^\diamond\{1\}$ is a direct summand of the graded properad $gr\mathcal{L}ie^\diamond\mathcal{B}_\infty$ associated with the genus filtration of the properad $\mathcal{L}ie^\diamond\mathcal{B}_\infty$. Hence the required result follows from the proof of Proposition 2.6.1. \square

We are interested in the operad $\mathcal{L}_\infty^\diamond$ because of the following property.

C.4.2. Lemma. *There is a monomorphism of dg operads,*

$$\chi : \mathcal{L}_\infty^\diamond \longrightarrow \mathcal{BV}_\infty^{\text{com}}$$

given on generators as follows,

$$\chi \left(\begin{array}{c} \textcircled{p} \\ | \\ \text{---} \\ / \quad \backslash \\ \dots \quad \dots \\ 1 \quad 2 \quad \dots \quad k-1 \quad k \end{array} \right) = \begin{array}{c} \boxed{p+k-1} \\ | \\ \text{---} \\ / \quad \backslash \\ \dots \quad \dots \\ 1 \quad 2 \quad \dots \quad k-1 \quad k \end{array}$$

Proof. For notation reason, we prove the proposition in terms of representations: for any representation $\rho : \mathcal{BV}_\infty^{\text{com}} \rightarrow \mathcal{E}nd_V$ we construct an associated representation $\rho' : \mathcal{L}_\infty^\diamond \rightarrow \mathcal{E}nd_V$ such that $\rho' = \rho \circ \chi$.

Let $\{\Delta_a : V \rightarrow V[1 - 2a], \mu : \odot^2 V \rightarrow V\}_{a \geq 1}$ be a $\mathcal{BV}_\infty^{\text{com}}$ -structure in a dg vector space (V, d) . Then

$$\Delta := d + \sum_{a \geq 1} \hbar^a \Delta_a$$

is a degree 1 differential in the graded vector space $V[[\hbar]]$, \hbar being a formal parameter of homological degree 2. As explained in §5.1, this differential makes the graded commutative algebra $(V[[\hbar]], \mu)$ into a $\mathcal{L}ie_\infty\{1\}[[\hbar]]$ algebra over the ring $\mathbb{K}[[\hbar]]$, with higher Lie brackets given by,

$$F_k^\Delta := \frac{1}{(k-1)!} \underbrace{[\dots [[\Delta, \mu], \mu], \dots, \mu]}_{k-1 \text{ brackets}} = \frac{\hbar^{k-1}}{(k-1)!} \sum_{p=0}^{\infty} \hbar^p \underbrace{[\dots [[\Delta_{p+k-1}, \mu], \mu], \dots, \mu]}_{k-1 \text{ brackets}}$$

It is well-known that $\mathcal{L}ie_\infty$ -algebra structures are homogeneous in the sense that if $\{\mu_n\}_{n \geq 1}$ is a $\mathcal{L}ie_\infty$ -algebra structure in some vector space, then, for any $\lambda \in \mathbb{R}$, the collection $\{\lambda^{n-1} \mu_n\}_{n \geq 1}$ is again a $\mathcal{L}ie_\infty$ -algebra structure in the same space. Therefore, the rescaled collection of operations,

$$\hat{F}_k^\Delta := \frac{1}{(k-1)!} \sum_{p=0}^{\infty} \hbar^p \underbrace{[\dots [[\Delta_{p+k-1}, \mu], \mu], \dots, \mu]}_{k-1 \text{ brackets}}$$

also defines a continuous representation of $\mathcal{L}ie_\infty\{1\}[[\hbar]]$ in the dg space $(V[[\hbar]], \Delta)$, and hence a representation of \mathcal{L}_∞° in the space (V, d) given on the generators as follows,

$$\rho' \left(\begin{array}{c} \circ \\ \diagup \quad \diagdown \\ 1 \quad \dots \quad k-1 \quad k \end{array} \right) = \frac{1}{(k-1)!} \underbrace{[\dots [[\Delta_{p+k-1}, \mu], \mu], \dots, \mu]}_{k-1 \text{ brackets}}$$

As $\rho \left(\begin{array}{c} \boxed{p+k-1} \\ \diagup \quad \diagdown \\ 1 \quad \dots \quad k-1 \quad k \end{array} \right)$ equals $\frac{1}{(k-1)!} \underbrace{[\dots [[\Delta_{p+k-1}, \mu], \mu], \dots, \mu]}_{k-1 \text{ brackets}}$ as well, the proof is completed. \square

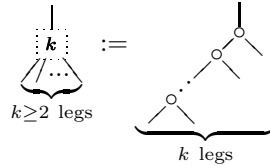
Finally we can give the proof of Theorem 5.3.1.

Proof of Theorem 5.3.1. The map π induces obviously a morphism of operads,

$$[\pi] : H^\bullet(\mathcal{B}\mathcal{V}_\infty^{com}) \longrightarrow \mathcal{B}\mathcal{V}.$$

Therefore to prove the theorem it is enough to show that $[\pi]$ induces an isomorphism of \mathbb{S} -modules.

Denote the following (equivalence class of a) graph in $\mathcal{B}\mathcal{V}_\infty^{com}$ by



and call it a *dashed square vertex*. Consider an operad, \mathcal{O} , freely generated by the operad $\mathcal{C}om$ and a countable family of unary operations, $\left\{ \begin{array}{c} \circ \\ | \\ \circ \end{array} \right\}_{a \geq 1}$ of homological degree $1 - 2a$ (so that $\mathcal{B}\mathcal{V}_\infty^{com}$ is a quotient

of \mathcal{O} by the ideal encoding the requirement that each unary operation $\begin{array}{c} \circ \\ | \\ \circ \end{array}$ is of order $\leq a + 1$ with respect to the multiplication operation). Let $t^{(1)}$ be any tree built from the following ‘‘corollas’’,

$$\begin{array}{c} \circ \\ | \\ \circ \end{array} \quad := \quad \begin{array}{c} \circ \\ | \\ \circ \\ | \\ \circ \end{array} \quad \text{with } k \geq 1, a \geq 1$$

(where we assume implicitly that for $k = 1$ the l.h.s. corolla equals $\begin{array}{c} \circ \\ | \\ \circ \end{array}$) and let $t^{(2)}$ be any graph obtained by attaching to one or more (or none) input leg of a dashed square vertex a tree of the type $t^{(1)}$, e.g.

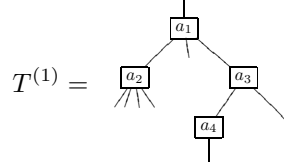
$$t^{(2)} = \begin{array}{c} \circ \\ | \\ \circ \\ | \\ \circ \\ | \\ \circ \end{array}$$

The family $\{t^{(1)}, t^{(2)}\}$ forms a basis of \mathcal{O} as an \mathbb{S} -module.

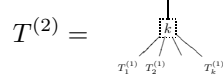
Define for any $a, k \geq 1$ a linear combination,

$$(47) \quad \begin{array}{c} \boxed{a} \\ \diagup \quad \dots \quad \diagdown \\ 1 \quad 2 \quad \dots \quad k-1 \quad k \end{array} = \begin{array}{c} \circledast a \\ \diagup \quad \dots \quad \diagdown \\ 1 \quad 2 \quad \dots \quad k \end{array} - \sum_{\sigma \in \mathbb{S}_k} \frac{1}{(k-1)!} \begin{array}{c} \boxed{a} \\ \diagup \quad \dots \quad \diagdown \\ \sigma(1) \quad \sigma(2) \quad \dots \quad \sigma(k-1) \end{array} + \sum_{\sigma \in \mathbb{S}_k} \frac{1}{2!(k-2)!} \begin{array}{c} \boxed{a} \\ \diagup \quad \dots \quad \diagdown \\ \sigma(1) \quad \sigma(k-1) \quad \sigma(k) \end{array} + \dots,$$

Consider next (i) a set $\{T^{(1)}\}$ of all possible trees generated by these “square” corollas, e.g.



and also (ii) a set $\{T^{(2)}\}$ of all possible trees obtained by attaching to (some) legs of a dashed square vertex trees from the set $\{T^{(1)}\}$, e.g.



Formulae (47) define a natural linear map

$$\phi : \text{span}\langle T^{(1)}, T^{(2)} \rangle \longrightarrow \text{span}\langle t^{(1)}, t^{(2)} \rangle = \mathcal{O}.$$

The expressions (47) can be (inductively) inverted,

$$(48) \quad \begin{array}{c} \circledast a \\ \diagup \quad \dots \quad \diagdown \\ 1 \quad 2 \quad \dots \quad k \end{array} = \begin{array}{c} \boxed{a} \\ \diagup \quad \dots \quad \diagdown \\ 1 \quad 2 \quad \dots \quad k-1 \quad k \end{array} + \sum_{\sigma \in \mathbb{S}_k} \frac{1}{(k-1)!} \begin{array}{c} \boxed{a} \\ \diagup \quad \dots \quad \diagdown \\ \sigma(1) \quad \sigma(2) \quad \dots \quad \sigma(k-1) \end{array} - \sum_{\sigma \in \mathbb{S}_k} \frac{1}{2!(k-2)!} \begin{array}{c} \boxed{a} \\ \diagup \quad \dots \quad \diagdown \\ \sigma(1) \quad \sigma(k-1) \quad \sigma(k) \end{array} + \dots,$$

and hence give us, again by induction, a linear map

$$\psi : \text{span}\langle t^{(1)}, t^{(2)} \rangle \longrightarrow \text{span}\langle T^{(1)}, T^{(2)} \rangle$$

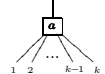
as follows. On 1-vertex trees from the family $\{t^{(1)}, t^{(2)}\}$ the map ψ is given by

$$\psi \left(\begin{array}{c} \boxed{k} \\ \diagup \quad \dots \quad \diagdown \\ \dots \quad \dots \end{array} \right) = \begin{array}{c} \boxed{k} \\ \diagup \quad \dots \quad \diagdown \\ \dots \quad \dots \end{array}, \quad \psi \left(\begin{array}{c} \circledast a \\ \diagup \quad \dots \quad \diagdown \\ 1 \quad 2 \quad \dots \quad k \end{array} \right) = \text{the r.h.s. of (48)}.$$

Assume that the map ψ is constructed on n -vertex trees from the family $\{t^{(1)}, t^{(2)}\}$. Let t be a tree with $n+1$ vertices. The complement to the root vertex of t is a disjoint union of trees, $\{t'\}$, with at most $n-1$ vertices. To get $\psi(t)$ apply first ψ to the subtrees t' to get a linear combination of trees, $\sum t''$, where each t'' is obtained by attaching to (some) input legs of a one-vertex tree v from $\{t^{(1)}, t^{(2)}\}$ an element of the set $\{T^{(1)}, T^{(2)}\}$; finally, apply ψ to the root vertex v of each summand t'' . By construction, $\psi \circ \phi = \text{Id}$ and $\phi \circ \psi = \text{Id}$ so that the map ϕ is an isomorphism,

$$\mathcal{O} \cong \text{span}\langle T^{(1)}, T^{(2)} \rangle.$$

The ideal I in \mathcal{O} defining the operad $\mathcal{BV}_\infty^{com}$ takes now a very simple form — this is an \mathbb{S} -submodule of \mathcal{O} spanned by trees from the family $\{T^{(1)}, T^{(2)}\}$ which contain at least one “bad” square vertex



with $a < k-1$. If $\{T_+^{(1)}, T_+^{(2)}\} \subset \{T^{(1)}, T^{(2)}\}$ is a subset of trees containing no bad vertices, then we can write an isomorphism of \mathbb{S} -modules,

$$\mathcal{BV}_\infty^{com} \cong \text{span}\langle T_+^{(1)}, T_+^{(2)} \rangle$$

Therefore, as a complex $\mathcal{BV}_\infty^{com}$ splits into a direct sum of subcomplexes,

$$\mathcal{BV}_\infty^{com} \cong \text{span}\langle T_+^{(1)} \rangle \oplus \text{span}\langle T_+^{(2)} \rangle.$$

The subcomplex $\text{span} \langle T_+^{(1)} \rangle$ is the image of the dg operad \mathcal{L}_∞° under the monomorphism χ (see Lemma C.4.2) so that we get eventually an isomorphism of complexes,

$$\mathcal{BV}_\infty^{\text{com}} \cong \text{Com} \circ \mathcal{L}_\infty^\circ,$$

and the above splitting corresponds to the augmentation splitting of the operad Com ,

$$\text{Com} = \text{span} \langle 1 \rangle \oplus \overline{\text{Com}}.$$

Therefore, by Proposition C.4.1 and Remark C.3,

$$H^\bullet(\mathcal{BV}_\infty^{\text{com}}) \cong \text{Com} \circ \mathcal{L}^\circ \cong q\mathcal{BV}.$$

By Proposition C.2, we get isomorphisms of \mathbb{S} -modules,

$$H^\bullet(\mathcal{BV}_\infty^{\text{com}}) \cong \text{Gr}(\mathcal{BV}) \cong \mathcal{BV}$$

which completes the proof of the Theorem. \square

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